DOCUMENT RESUME

ED 264 088 SE 046 235

AUTHOR	Usiskin, Zalman; Bell, Max
TITLE	Applying Arithmetic: A Handbook of Applications of Arithmetic. Part II: Operations.
INSTITUTION	Chicago Univ., Ill. Dept. of Education.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	83
GRANT	NSF-SED-79-19065
NOTE	225p.; For related documents, see SE 046 234 (part I) and SE 046 236 (part III).
PUB TYPE	Guides - Classroom Use - Guides (For Teachers) (052)
EDRS PRICE	MF01/PC09 Plus Postage.
DESCRIPTORS	Addition; *Arithmetic; Division; Elementary Education; *Elementary School Mathematics; *Mathematical Applications; Mathematics Education; *Mathematics Instruction; Multiplication; Subtraction; *Teaching Methods
IDENTIFIERS	*Exponents (Mathematics); National Science Foundation

ABSTRACT

Chapters 5 through 10 of a 14-chapter, three-volume work on arithmetical applications are contained in this document. Each chapter details the "use classes" of one broad arithmetical concept. (A "use class" of a concept is a set of examples of real world uses of the concept which share a common structure). Each chapter contains: an introduction and summary; three to six sections, each devoted to one use class and containing a general introduction, questions, and comments; suggestions for teaching or illustrating a given concept; questions which test understanding of the ideas presented; and notes and commentary, with reasons for selecting particular use classes, related research, and short essays on issues related to applying the concepts. Topics of the chapters include: uses of addition, discussing putting together, shift, and addition from subtraction (chapter 5); uses of subtraction, discussing take-away, comparison, subtraction shift, and recovering addend (chapter 6); uses of multiplication, considering size change, acting across, and rate factor (chapter 7); uses of division, considering ratio, rate, rate divisor, size change divisor, and recovering factor (chapter 8); uses of powering, discussing change of dimension, growth, and notation (chapter 9); and uses that combine operations, such as those involving exactly two classes (chapter 10). (JN)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

U.S. DEPARTMENT OF EDUCATION
NATIONAL INSTITUTE OF EDUCATION
EDUCATIONAL RESOURCES INFORMATION
CENTER (ERIC)

☑ This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

- Points of view or opinions stated in this document do not necessarily represent official NIE position or policy.

"PERMISSION TO REPRODUCE THIS
MATERIAL HAS BEEN GRANTED BY
Zalman Usiskin
Max Bell
TO THE EDUCATIONAL RESOURCES
INFORMATION CENTER (ERIC)."

APPLYING ARITHMETIC

A HANDBOOK OF APPLICATIONS OF ARITHMETIC

PART II: OPERATIONS

by

ZALMAN USISKIN AND MAX BELL

under the auspices of the
ARITHMETIC AND ITS APPLICATIONS PROJECT

DEPARTMENT OF EDUCATION
THE UNIVERSITY OF CHICAGO

The preparation of this work was partially supported by National Science Foundation Grant SED 79-19065. However, any opinions, conclusions, or recommendations expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Copyright ©1983 by The University of Chicago

Preface

The writing of this volume has been motivated by two existing gaps in mathematics education. The first gap is between student performance on arithmetic skills and the generally worse performance on realistic problems utilizing those same skills. The second gap is the disparity between oft-stated goals of professional organizations and schools and textbooks (generally supportive of applications of arithmetic) and the classroom reality. After grade 4, realistic applications of arithmetic do not often appear in the classroom, and those that do appear represent only a narrow picture of this broad domain.

The intended audiences are diverse. First, we have designed the book for use by teachers. Each concept is illustrated by a large number of examples, and comments are given following the examples to aid in adaptation for use in classrooms. Each chapter contains a special section entitled "Pedagogical Remarks" to further assist in this task.

Second, we have designed the book for use by those interested in curriculum design or research. Each chapter contains an extended discussion of selected theoretical, pedagogical, philosophical, psychological or semantic issues and research related to the ideas found within the chapter.

Third, because teachers and other professional educators often encounter books like this one only in the context of coursework, we have included a number of questions at the end of each chapter.

Fourth, we hope that the ideas in this book might also be suitable to lay readers interested in understanding the uses of arithmetic. We have tried to make the writing easy to understand and in most places the mathematical prerequisites necessary to comprehend the material are minimal.

Our goal is to improve our society's understanding of the applications of arithmetic. In the past, due to the necessity of having to spend a great deal of time teaching how to get answers, books could not afford to be devoted to teaching when to use particular arithmetic processes. Calculators, in our opinion, allow us to change emphasis from how to when. This book constitutes a first attempt to provide a rather complete categorization of the simpler applications of arithmetic.

The organization of this book is not definitive and in many places may not exhaust the range of applications. Many may disagree with our categorizations. We encourage criticism; we only hope that those who criticize will help us improve the ideas presented here or produce their own improved version.

Zalman Usiskin and Max Bell
June, 1983

Acknowledgements

This book was written as part of the Arithmetic and Its Applications project funded by the National Science Foundation. We are grateful to Ray Hannapel, Ruth von Blum, Harold Stolberg, Andrew Molnar, and others in the foundation for their support and assistance.

The Arithmetic and Its Applications project was assisted in its work by two advisory boards, one consisting of university personnel, the other of junior high school or middle school teachers and supervisors. The advisory board members were: Pamela Ames, Harry Bohan, Sherye Garmony, Alan Hoffer, Jeremy Kilpatrick, James McBride, Kay Nebel, and Jane Swafford. Roberta Dees worked with us on this project for a year. Each of these people assisted in the development of this manuscript in his or her own way (but the authors take full responsibility for the writing).

Early drafts of these materials were tried out in classes at the University of Chicago by us, Sam Houston State University by Harry Bohan, and Ohio State University by Alan Hoffer. We appreciate the willingness of these institutions to support this endeavor and extend our thanks to the students who gave comments to help us improve it.

Our thanks go also to the University of Chicago for providing facilities, colleagues, and students particularly amenable to the kind of thinking this type of writing requires.

Finally, we are each fortunate to have wives who are not only supportive of our work but who also are involved in mathematics education. They have been responsive sounding boards for most of the ideas presented here and often were the ones who provided an ultimate clarification of an issue. We appreciate their help more than we can put in words.

Zalman Usiskin and Max Bell

Introduction

For most of us, an application of arithmetic begins with an attempt to comprehend numbers we encounter in everyday living. These range from prices of goods and services to interest rates on investments to sports scores to minimum daily nutrient requirements to ID numbers to geographic information found on roads and maps to technical information about objects around the home to results of surveys published in newspapers or magazines. Our society has become increasingly numericized, requiring each of us to process more numbers than many of us thought we would need.

On many occasions, comprehension of numerical information suffices. We only may want to know the protein content of a food, or a sports score, or the time to the airport, or a social security number. At other times we may wish to operate on given numerical information to generate more information. From prices of foods, one may calculate which is more economical and still supply nutritional needs. From interest rates, income can be determined. From temperature data, energy costs can be estimated. From sports data, decisions regarding the quality of teams and participants may be desired. From information about the size of living quarters, wall and floor covering needs can be established. We add, subtract, multiply, divide, take powers, and apply other operations of arithmetic to help us obtain the additional numerical information.

But things are not always so simple. Given numerical information is not always written in a form that makes it easy to operate upon.

BEST COPY AVAILABLE

We do not always know what to do with such information until we display, scale, or estimate it in some way. We classify rewriting, graphing, scaling, and estimating as maneuvers and recognize that we often maneuver both given numerical information and the results of computations.

These three application skills comprise the subject matter of this book, one part of the book being devoted to each of them.

In Part I, Numbers we ask To what uses are numbers and number aggregates put?

In Part II, Operations we ask What are the common uses of the fundamental operations?

In Part III, Maneuvers we ask For what reasons are the most common types of maneuvers applied?

The three parts are divided into a total of 14 chapters. Each chapter details the use classes of one broad arithmetic concept (e.g., single number, multiplication, or estimation). The notion of use class is at the heart of this book and is roughly defined here.

> A use class of a concept is a set of examples of real world uses of the concept which share a common structure.

The arithmetic concepts in this book have from 3 to 6 use classes each; there are 57 use classes in the 14 chapters. The chapters are organized in the following way.

Introduction

3-6 Sections, one devoted to each use class, each
with a general introduction to that class followed
by example questions with answers and comments

Summary

Pedagogical Remarks

Questions

Notes and Commentary

BEST COPY AVAILABLE

Since use classes are defined in terms of examples, the major space in this volume is devoted to the examples, answers, and comments. The purpose of the other components of each chapter is as follows:

The introduction and summary each contain a short synopsis of the types of applications of a particular concept.

Suggestions for teaching or illustrating a given concept may be found both in the comments following each example and in the pedagogical remarks.

The questions are a test of the reader's understanding of the ideas herein. The notes and commentary include our reasons for the selection of the particular use classes, related research, and short essays on issues related to applying the various concepts.

A calculator is strongly recommended for all sections of this book so that the reader can spend time dealing with the concepts of this book rather than with paper and pencil computation. A calculator with an x^y key is necessary in Chapters 9 and 10.

APPLYING ARITHMETIC

Table of Contents

Part I: Numbers 2

Chapter 1: Uses of Single Numbers 5

A. Counts 6
B. Measures 9
C. Locations 16
D. Ratio Comparisons 25
E. Codes 30
F. Derived Formula Constants 34

Summary 38
Pedagogical Remarks 39
Questions 47
Notes and Commentary 50

Chapter 2: Uses of Ordered Pairs, Triples, or n-tuples 57

A. Counts 59
B. Measures 61
C. Locations 63
D. Ratios 66
E. Codes 68
F. Combined Uses 70

Summary 72
Pedagogical Remarks 73
Questions 76
Notes and Commentary 78

Part I: Numbers (continued)

Chapter 3: Uses of Collections of Numbers 83

A. Domains 85
B. Data Sets 87
C. Neighborhoods 91
D. Solution sets 95

Summary 98
Pedagogical Remarks 99
Questions 102
Notes and Commentary 104

Chapter 4: Uses of Variables 109

A. Formulas 110
B. Unknowns 116
C. Properties 120
D. Storage Locations 124

Summary 128
Pedagogical Remarks 129
Questions 134
Notes and Commentary 137

Summary of Part I 141

Part II: Operations 143

Chapter 5: Uses of Addition 149

A. Putting Together 150
B. Shift 156
C. Addition from Subtraction 161

Summary 164
Pedagogical Remarks 165
Questions 167
Notes and Commentary 168

Chapter 6: Uses of Subtraction 177

A. Take-away 178
B. Comparison 182
C. Subtraction Shift 187
D. Recovering Addend 189

Summary 191
Pedagogical Remarks 192
Questions 195
Notes and Commentary 197

Part II: Operations (continued)

Chapter 7: Uses of Multiplication 203

- A. Size Change 204
- B. Acting Across 214
- C. Rate Factor 221

- Summary 227
- Pedagogical Remarks 228
- Questions 233
- Notes and Commentary 235

Chapter 8: Uses of Division 245

- A. Ratio 246
- B. Rate 251
- C. Rate Divisor 257
- D. Size Change Divisor 260
- E. Recovering Factor 264

- Summary 267
- Pedagogical Remarks 268
- Questions 273
- Notes and Commentary 275

Summary of the Use Classes of the Four Fundamental Operations 283

Chapter 9: Uses of Powering 285

- A. Change of Dimension 289
- B. Growth 295
- C. Notation 301

- Summary 308
- Pedagogical Remarks 309
- Questions 314
- Notes and Commentary 315

Chapter 10: Uses that Combine Operations 323

- A. Applications Involving Exactly Two Use Classes 325
- B. Applications Involving More Than Two Use Classes 332
- C. Applications Not Readily Separable into Constituent Use Classes 337

- Summary 342
- Pedagogical Remarks 343
- Questions 347
- Notes and Commentary 349

Summary of Part II 353

Part III: Maneuvers 354

Chapter 11: Reasons for Rewriting 357

A. Constraints 359
B. Clarity 363
C. Facility 370
D. Consistency 375

Summary 378
Pedagogical Remarks 379
Questions 383
Notes and Commentary 385

Chapter 12: Reasons for Estimating and Approximating 391

A. Constraints 393
B. Clarity 399
C. Facility 403
D. Consistency 406

Summary 410
Pedagogical Remarks 411
Questions 416
Notes and Commentary 419

Chapter 13: Reasons for Transforming 425

A. Constraints 427
B. Clarity 431
C. Facility 434
D. Consistency 438

Summary 442
Pedagogical Remarks 443
Questions 446
Notes and Commentary 451

Chapter 14: Reasons for Displaying 453

A. Constraints 455
B. Clarity 461
C. Facility 471
D. Consistency 477

Summary 483
Pedagogical Remarks 484
Questions 487
Notes and Commentary 492

Summary of Part III 496

Postscript 497

Bibliography 500

Index 505

PART II: OPERATIONS

"You subtract when there are two large numbers in the problem. You add if there are more than two numbers. If there is a large number and a small one, you divide if it will come out even. If it won't come out even, you multiply." (P. R. Stevenson, _Journal of Educational Research_, XI (1917), pp. 95-103.)

Introduction

Consider calculations arising from real problems, like 40 km - 17 cm, or $\frac{3}{2} \times \$6.19$, or $(1.07)^4$. In Part I, we have presented what the numbers in these calculations might represent or quantify from the real world. The purpose of this part of the handbook is to display the ways in which the operations are used and thus to provide a framework for teaching students when to use each operation.

Within the context of the teaching of arithmetic in schools, this part may be viewed as an elaboration of the skill "choosing an operation". We believe that students often have trouble with problems of that type because they are given little more help than intuition or taught fixed rules, such as looking for key words, that are not always reliable.

In this regard, the major pedagogic blunder is to mislead students into thinking that each fundamental operation has only one kind of use. For example, to teach subtraction only as take-away is not just an oversimplification, it keeps the student from understanding why subtraction is found in so many situations that are not "take-away" situations and makes it difficult for the student to learn other uses. A major reason why the operations of addition, subtraction, multiplication, division and powering are considered fundamental is because each has more than one important kind of use.

The organization of Part II is similar to that of Part I. Each operation has a chapter devoted to its uses. A sixth chapter is devoted to a brief introduction to uses that combine operations.

Within the first five chapters, use classes for each operation are described and exemplified. The use classes are of two types, those use classes directly identified with use meanings of the operations and those use classes derived in some mathematical way from the use meanings of that or a related operation. Though specifics are given throughout the chapters, we explain the broad ideas here.

Use classes from use meanings. A use meaning of an operation is a fundamental way in which the operation is used. The use meaning is so closely associated with the operation that the meaning can be used to teach the operation just as often as the operation can help students to understand the meaning. The use meanings are listed here; several of these meanings are commonly taught in arithmetic and most will be familiar to the reader.

Operation	Use Meanings
Addition	putting together shift
Subtraction	take-away comparison
Multiplication	size change acting across
Division	ratio rate
Powering	change of dimension growth

Use classes derived from use meanings via mathematical relationships among the operations. The operations are related to each other in ways which cause a use meaning of one operation to be converted into a use requiring another operation. For example, the following problem involves the take-away use meaning but does not require subtraction for its answer.

> After spending $3.25 for lunch, a person
> has $4.63 left. How much did the person
> have before lunch?

While we might classify the problem as an instance of the "put together" use meaning of addition, we also recognize that it is derived from the take-away use meaning of subtraction by the related facts relationship: $b = c - a$ implies $a + b = c$.

Related facts relationships are taught by many teachers to help students learn basic facts and check subtraction problems. We emphasize here that such relationships are important also for applying the operations in problems. Four kinds of relationships lead to new use classes. They are listed on the next page. At this point, the reader is not expected to understand how these relationships connect with use classes. The particulars are given as each use class is described.

Mathematical Relationships that Give Rise to Use Classes

Relationship Name	Numerical Example	Symbolic Description*	Operations Related
Related facts	$2 + 3 = 5$, so $2 = 5 - 3$ and $3 = 5 - 2$.	$a + b = c$ iff $a = c - b$. $a + b = c$ iff $b = c - a$.	addition, subtraction
	$6 \times 50 = 300$, so $6 = 300/50$ and $50 = 300/6$.	$a \times b = c$ iff $a = c \div b$. $a \times b = c$ iff $b = c \div a$.	multiplication, division
	$2^9 = 512$, so $2 = 512^{1/9}$.	$a^b = c$ iff $a = \sqrt[b]{c} = c^{1/b}$	powering with self
Inverse operations	$6 - 7.5 = 6 + -7.5$.	$a - b = a + -b$.	addition, subtraction
	$5 \div 3/4 = 5 \times 4/3$.	$a \div b = a \times 1/b$.	multiplication, division
Repetition	$\underbrace{-2 + -2 + -2}_{n\ \text{terms}} = 3 \times -2$.	$\underbrace{a + a + \ldots + a}_{n\ \text{terms}} = na$.	addition, multiplication
	$\underbrace{1.4 \times 1.4 \times 1.4}_{n\ \text{factors}} = 1.4^3$.	$\underbrace{a \times a \times \ldots \times a}_{n\ \text{factors}} = a^n$.	multiplication, powering
Double reverse	$100 - 97 = 3$, so $100 - 3 = 97$.	$a - b = c$ iff $a - c = b$.	subtraction with self
	$56 \div 8 = 7$, so $56 \div 7 = 8$.	$a \div b = c$ iff $a \div c = b$.	division with self

*Iff is the abbreviation for "if and only if" and indicates that the equality on either side of the "iff" gives rise to the other side. For example, in the first related facts relationship, just as $2 + 3 = 5$ gives rise to $2 = 5 - 3$, so if you know that $5 - 3 = 2$, it gives rise to $2 + 3 = 5$.

CHAPTER 3

USES OF ADDITION

Addition has two basic use meanings: putting together and shift. These meanings tend to be well taught in early school work, particularly with whole numbers, often with concrete experiences that are neglected in teaching other operations.

From its related facts relationship with subtraction, addition obtains one derived use meaning. Thus we separate this chapter into three sections:

A. Putting together

B. Shift

C. Addition from subtraction

Addition Use Class A: Putting Together

Putting together is the first action that children learn to associate with an operation. If three objects are placed next to four other objects, then there are seven objects in all. We add to find the total. Instances of this use class do not require that things physically be placed together; they may only be considered together. For example, if there are 270 students in one school and 312 students in another, then there are 582 students in all.

This use class has a wider range of applicability than is often seen in school work. For example, there are geometric applications: If two stoplights are 1.3 miles apart and the next stoplight is a half mile further, then the first and third stoplights are 1.8 miles apart. There are also instances involving percentages and probabilities, as the examples show.

Examples:

1. Total count. In the U.S. Congress, there are 435 members of the House of Representatives and 100 members of the Senate. How many people are there in all?

   Answer: 435 + 100 = 535.

   Comment: It is correct to write
   435 people + 100 people = 535 people
   to describe what it is that is being counted.

2. Total count. In 1973 in the United States, 57,400,000 tires were installed as original equipment on new cars, and 150,000,000 tires were sold as replacements for old tires. How many tires were put on cars in 1973?

   Answer: 57.4 million + 150 million = 207.4 million.

   Comment: Decimals are often used with word names to denote large numbers, but we seldom see this usage in schoolbooks.

3. Money. With $20, a person wants to buy a shirt for $10.95 and a pair of shorts for $8.95. Can this be done?

   Answer: The total to be paid is $10.95 + $8.95, or $19.90. That leaves just ten cents and in most states, the sales tax would be more than ten cents so both things probably could not be purchased with the $20.

   Comment: This kind of problem can be used to teach inequality: Is $\$10.95 + \$8.95 \leq \$20$?

4. Money. To buy a record album, three friends pool their resources of $1.23, $1.58, and $2.47. How much do they have altogether?

   Answer: $1.23 + $1.58 + $2.47 = $5.28

   Comment: It is well known that students can do computations with money before they are taught decimal notation. Thus money is a suitable vehicle for teaching decimals, and this can be done earlier than most books attempt it.

5. Length. To make shelves to fit odd spaces, boards of lengths 2'3", 2'5", and 2'7" are needed. What is the minimum length of a single board that would serve for the task?

   Answer: 7'3", so an 8-foot board should be purchased.

   Comment: The calculation is more difficult here than with money or metric units even though the numbers are very simple. This is a major reason for the move to the metric system.

6. Distance. A map shows that it is about 181 miles from Chicago to Indianapolis and about 111 miles from Indianapolis to Louisville, both distances measured along connecting Interstate highways. About how far is it from Chicago to Louisville along this route?

   Answer: 181 miles + 111 miles = 292 miles.

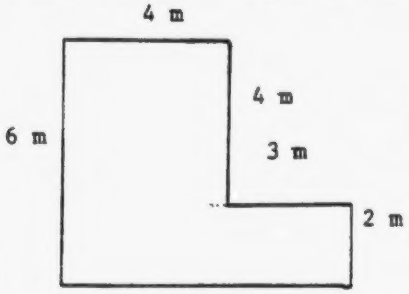
   Comment: One often adds travel times rather than distances. It is about $3\frac{1}{2}$ hours driving time from Chicago to Indianapolis...

7. Area. North America (including Central America) has an area of approximately 9,390,000 sq mi. South America has an area of about 6,795,000 sq mi. Approximately what is the total land area of the Americas?

Answer: 16,185,000 sq mi.

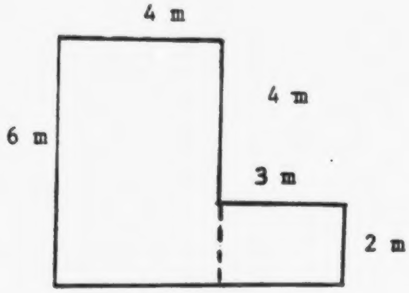
Comment: By itself this is not much of a problem. But it would be interesting to ask students to compare (by division or subtraction) the total with the area of Asia (16,988,000 sq mi), or (by division) with the total land area of the Earth (approximately 56,000,000 sq mi).

8. Area. What is the area of the living room below?



Answer: 30 m^2

Comment: The answer may be found by splitting the living room into two rectangles as shown, and adding the areas of the rectangles.



9. Diet. Here is what I ate yesterday; how many calories and how many grams of protein did I have at each meal?

Sample Menu	Calories	Grams of Protein
Breakfast		
1 slice of white bread, toasted, with butter and jam	167	2
1/2 cup of orange juice	55	0
1 scrambled egg	110	6
Lunch		
1/2 cup creamed cottage cheese	120	15
1 apple	70	0
1 doughnut	135	2
1 glass of cola	137	0
Dinner		
1/4 pound hamburger	327	28
on a bun	115	3
1/2 cup frozen peas	34	2
20 pieces French fried potatoes	310	2
1 cup whole milk	165	8

Answers:

	Calories	Protein
Breakfast	332	8
Lunch	462	17
Dinner	951	43

Comment: It is said that a person can utilize only 20 grams of protein in a 5-hour period; the excess is excreted. If so, how many grams of protein were wasted by this menu? (Answer: 23, since 43 - 20 = 23.) Furthermore, a person needs about 20 grams of protein in the morning to "get going" and about 60 grams in a day. So a good problem is to rearrange the food above to give a more equal 20-20-20 distribution of protein. (One way to do this is to move milk from dinner to breakfast.)

Comment: Many books give this information for common foods. Students can determine how many calories and grams of protein were in what they ate.

10. Probabilities. In the game Monopoly, a player is situated 4 spaces, 5 spaces, and 7 spaces from hotels. What is the probability of this player landing on hotels on the next turn?

Answer: In Monopoly, a player throws two dice and uses the sum as the number of spaces to move.

A player could land on hotels by throwing

(a) one sum of 4

(b) one sum of 5

(c) one sum of 7

(d) one sum of 2 ("doubles", forcing a second toss) followed by a sum of 2

(e) a sum of 2 followed by a sum of 3

(f) a sum of 2 followed by a sum of 5.

The probabilities of these are: (a) 3/36 or 1/12; (b) 1/9; (c) 1/6; (d) 1/36 x 1/36 or 1/1296; (e) 1/36 x 1/18, or 1/648; and (f) 1/36 x 1/9, or 1/324.

Adding these fractions gives a sum of 475/1296, or approximately .367.

Comment: Notice how little is added by the possibility of doubles.

11. Percentages. In the province of Manitoba, 26% of the people belong to the United Church of Christ, 25% are Roman Catholic, 12% are Anglican, 7% Lutheran, 6% Mennonite, and 6% Ukrainian Catholic. Assuming no one belongs to two churches, what percentage of the population of Manitoba do these religions cover?

Answer: Adding the percentages gives 82%.

Comment: Some people do belong to two different churches, so the sum is probably slightly less.

Comment: Examples 10 and 11 involve the addition of numbers used as ratio comparisons. See note 13.

12. Crosstabulations. A table of sex by grade in a middle school was typed incorrectly in a report. Where is the probable error?

	6	7	8	Total
Male	54	50	57	161
Female	49	48	63	150
Total	103	98	110	$\boxed{311}$

Answer: Adding across, there is an error in the Female row. Adding down, there is an error in the 8th grade column. Thus the most probable error is the number 63, which should probably be 53.

Comment: This kind of table, a crosstabulation, is often checked by the procedure used to get this answer.

Addition Use Class B: Shift

The temperature is 4° below zero and increases 5°. The resulting temperature can be calculated by doing the addition

$$-4° + 5°$$

yielding a sum of 1°. We call such additions shifts.

Shift use meaning of addition

Initial state + shift = final state

The shift use meaning conceptually differs from putting together in that the "shift" need not be a measure of a quantity, but a measure of change. For instance, in the temperature example, no 5° temperature quantity is involved in the situation.

Shifts need not involve negative numbers; the given temperature above could have been 40°, and have increased 5°, and would still be classified as a shift addition.

Shifts can represent either an increase or decrease from the initial state. A shift up, forward, or ahead is usually achieved by adding a positive number. A shift down, backward, or behind may be achieved either by adding a negative number (considered as an instance of this use class) or by subtracting a positive number (considered as an instance of a derived use class of subtraction--see Section C of Chapter 6). Adding a negative is used when one wants to specify a single operation which will work in both directions; this is almost always the case in formulas and in computer programs because it is easier to change the number inputs than the operations.

One shift can follow another, and the situation can then be thought of in either of two ways:

A. (initial state + shift) + shift

B. initial state + (shift + shift)

For situations of type B, the problem may become adding two shifts, with the initial state ignored. Such situations are found in Examples 8-10.

Examples:

1. Football. A quarterback has gained 354 yards for the season and gains 6 yards on the next play. How many yards has the runner gained for the season after this play?

   Answer: 354 yards + 6 yards = 360 yards.

   Comment: Suppose the runner now loses 5 yards on the next play. What then will be his season total yardage? The answer is
   360 yards + -5 yards = 355 yards.

   Comment: This is not exactly "putting together" addition because the addend at left is a total and in some sense of a different quality than the addend at right, the shift. The key quality of this example is the initial state (total so far) and a gain or loss from that state. Hence we classify this as shift rather than as putting together.

2. Test scores. The scores on a test are so disastrous that the instructor decides to add 15 points to each score so that she can use her usual grading scale. If a student originally had a score of 68, what is that student's new score?

   Answer: 83.

3. Golf. A golfer is 2 below par and gets a birdie on the next hole. Relative to par, what is presently the golfer's score?

   Answer: Relative to par, the golfer's score is now -2 + -1 or -3, meaning 3 below par.

Comment: For a score on a given hole, the phraseology in golf related to par and addition shifts is as follows:

double eagle	3 below par	add -3
eagle	2 below par	add -2
birdie	1 below par	add -1
par	even par	add 0
bogey	1 above par	add 1
double bogey	2 above par	add 2
triple bogey	3 above par	add 3

4. Sales. Three thousand dollars behind a quota, a salesperson makes a sale of $4000. How is the salesperson now doing relative to the quota?

Answer: -$3000 + $4000 = $1000.

Comment: The use of negative numbers helps in understanding situations like this one.

5. Length. Susan is 130 cm tall. Her older brother is 4 cm taller. How tall is he?

Answer: 130 cm + 4 cm = 134 cm.

Comment: Here Susan's height is the "initial state"; her brother's height is the "final state". Some people are uncomfortable classifying this as a shift because Susan's height is not changed in any way. The more classic shift would be as follows: Susan is 130 cm tall and grows 4 cm. How tall is she now? Conceptually the use meaning is the same, though the "feel" is slightly different.

6. Directed distances. A ship is 400 miles east of a checkpoint and travels west 30 miles. Where is the ship now in relation to the checkpoint?

Answer: With east as positive, the ship is now

400 + (-30) miles from the checkpoint.

This sum is 370, indicating 370 miles east of the checkpoint.

Comment: Notice that if the ship keeps travelling west, the ship will ultimately pass the checkpoint going west; thereafter the ship would be negative miles, or miles west, from the checkpoint.

7. Stock Market. A stock closed at $13\frac{1}{2}$ one day and went up $\frac{3}{4}$ the next. What is its new price?

Answer: $13\frac{1}{2} + \frac{3}{4} = 14\frac{1}{4}$.

Comment: In newspapers, changes downwards are indicated by negatives. The unit, not usually stated or written by those who deal with stocks, is dollars per share.

8. Stock Market. A stock goes up $\frac{1}{8}$ (dollars per share) one day and down $\frac{3}{8}$ the next. What is the total change?

Answer: $\frac{1}{8} + -\frac{3}{8} = -\frac{2}{8}$.

Comment: Of course one could subtract $\frac{3}{8}$ from $\frac{1}{8}$ to get the answer. This flexibility exemplifies the inverse relation between addition and subtraction, that subtraction is "adding the opposite."

Comment: Suppose the original price of the stock was $18\frac{3}{4}$. Then to get the final price, one has two additions:

$$18\frac{3}{4} + \frac{1}{8} + -\frac{3}{8}.$$

The left addition is of the form (original + shift) illustrated in earlier examples; the right addition is of the form (shift + shift).

9. Football. A football team loses 3 yards on one play and loses 4 yards on the next. What is the total loss on the two plays?

Answer: 7 yards.

Comment: No one would do this problem by adding

$$-3 \text{ yd} + -4 \text{ yd to get } -7 \text{ yd},$$

but if there were gains and losses together it would be efficient to use positive numbers for the gains and negative numbers for losses.

10. Angles. A quarter turn of one face of Rubik's cube is followed by a half turn of the same face in the same direction. What is the total amount of turn?

Answer: $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

Comment: The unit here, unstated, is revolutions.

Comment: Rotations and turns may be measured in revolutions, degrees, radians, and grads.

1 revolution = 360 degrees = 2π radians = 400 grads.

If the second turn had been in the opposite direction, one would add $\frac{1}{4} + -\frac{1}{2}$, resulting in $-\frac{1}{4}$, a quarter turn in that opposite direction.

Addition Use Class C: Addition from Subtraction

Every subtraction fact converts to an addition fact because of the related facts relationship between these operations: If a - b = c, then a = c + b. (For example, since 100 - 3 = 97, 97 + 3 = 100). Similarly, any subtraction situation can be converted into an addition situation merely by switching known and unknown information. For example, here is a problem that feels like a take-away or shift subtraction.

> After a $50 discount, a TV set is selling for $369.95. What was the original price?

As a subtraction problem, here is how this problem is written:

original price - $50 = $369.95.

Yet to get the answer one adds the given numbers.

original price = $369.95 + $50

= $419.95.

Many people learn to treat this and similar problems as addition without going through the subtraction. Mathematically, since addition is the operation applied to the given numbers, this is the more efficient approach. One can get away without thinking of subtraction here, but not without doing addition.

We say that problems of this type constitute a derived use class for addition, i.e., a use class not based on a use meaning of addition but instead based upon a use meaning of another operation. We call this use class addition from subtraction.

Examples:

1. Length. A piece $3\frac{1}{4}$" long was cut off a board, leaving a smaller board whose length is now $3'8\frac{11}{16}$". How long was the original?

   Answer: Original - $3\frac{1}{4}" = 3'8\frac{11}{16}"$, so

   $3'8\frac{11}{16}" + 3\frac{1}{4}" = 3'11\frac{15}{16}"$,

   what seems to be the length of the original. The actual length was probably 4', with the saw cut taking $\frac{1}{16}$".

   Comment: "Cut off" suggests take-away subtraction. Thus semantic cues do not always indicate the correct operation to use.

2. Money. Having spent \$1.83 for lunch, Jane has \$2.27 left. How much did she have before lunch?

   Answer: (Before lunch) - \$1.83 = \$2.27,

   so she had \$2.27 + \$1.83 or \$4.10.

   Comment: This problem can be extended by having Jane purchase more than one item, yielding more subtractions to convert to addition.

3. Scores. After being penalized 10 points for handing a paper in late, a student received a grade of 76. What would have been the grade had there been no penalty?

   Answer: original - penalty = final.

   original - 10 = 76.

   So original = 76 + 10 = 86.

   Comment: The subtraction could be considered either take-away or subtraction shift.

4. Profits. In business, profit = selling price - cost. (That is, p = s - c.) If a piece of merchandise costs \$10.26 and a \$5.60 profit is desired, what should be the selling price?

Answer: From $p = s - c$

$p + c = s$

So the selling price should be at least \$10.26 + \$5.60, or \$15.76.

Comment: All formulas involving subtraction have equivalent forms involving addition.

Summary

The chart summarizes the three use classes of addition discussed in this chapter.

Use class	Origin	a	b	a + b
putting together	use meaning	quantity	quantity	total quantity
shift	use meaning	initial state	shift	final state
addition from subtraction	derived from subtraction via related facts	final state or final quantity	amount of shift, amount taken away	initial state or original quantity

Children normally do problems categorizable into each of these use classes in their study of arithmetic, but textbook problems tend to cover only a narrow range of number situations.

Pedagogical Remarks

Use meanings vs. other kinds of meaning. Use meanings are only one type of meaning that can be given to an operation (see note 5 under Notes and Commentary for this chapter). The astute user of arithmetic knows a variety of meanings for operations. Instruction that stresses only use meanings would be as inadequate as present instruction that neglects them. Meanings of operations that come from knowing basic facts, or being able to do algorithms, or from working with structural properties, or from manipulating concrete materials, all contribute to a person's overall understanding of the operations.

Putting together. No use of mathematics is as well taught (and as well understood) as this use meaning of addition. Even so, uses of large numbers are neglected. Utilize wide variety in examples and take advantage of calculators. Almanacs are a particularly rich source of data.

Shift. Many students will want to use addition only for upward shifts and will want to use subtraction for downward shifts. This desire is fine; in fact, we have a use class of subtraction called "subtraction shift", but flexibility is important here. One way to encourage the use of addition both ways is to set up situations like the following, where the change can be in either direction:

Yesterday's stock price + change = today's price.

Amount ahead or behind some quota + today's result = amount ahead or behind quota at the end of day.

Number of students in the school at beginning of month + change in number of students = number of students in the school at end of month.

Addition from subtraction. The related facts relationship between addition and subtraction is very important and useful for remembering basic facts, for checking answers to subtraction problems, and ultimately for solving equations in algebra. When a subtraction problem is completed, you may wish to ask students to give the corresponding addition problem. Then, when covering any of the use meanings of subtraction, ask students to change the given information and the answer to make up a corresponding situation for addition. The skill of finding related facts is important for addition uses, and has big payoffs as well in the corresponding relationship that exists between multiplication and division.

Questions

1. Make up a putting together addition problem utilizing quantities of each of the following types: (a) land areas, (b) energy usage of some kind, (c) attendance at concerts or sporting events.
2. Make up a situation, different from the one in this chapter, in which addition of probabilities is appropriate.
3. Convert the following subtraction situation into an addition situation by answering the question and then switching some known and unknown information. Situation: 6 dozen cookies are baked and 5 cookies are eaten. How many remain?
4. Follow the price of a certain stock in a newspaper for a week, each day indicating the closing price and the amount of change from the previous day. Explain how, by adding the changes, one can check all of the arithmetic at once.
5. What does a shift of 0 mean?
6. Name one similarity and one difference between a shift of _s_ and a shift of _-s_.
7. A small shoe store reports the following sales for a week:

Day	Men's shoes	Women's shoes	Children's shoes
Monday	6 pair	15 pair	10 pair
Tuesday	3 pair	11 pair	7 pair
Wednesday	8 pair	11 pair	8 pair
Thursday	10 pair	17 pair	10 pair
Friday	4 pair	12 pair	10 pair
Saturday	16 pair	31 pair	48 pair

In what totals might the owner of this shoe store be interested?

8. A person willed 1/3 of his estate to his wife, 1/4 to his only son, 1/5 to his only daughter, and 1/6 to his business partner. (a) Is this possible? (b) If so, was there any left? If not, why not?

Notes and Commentary

Notes 1-8 relate to all of Part II.

1. The general context
2. The classification process
3. Classification criteria
4. Classification non-criteria
5. Other kinds of meanings for operations
6. The work of Sutherland
7. The works of Vest and Kansky
8. Work of others

Notes 9-16 relate specifically to this chapter.

9. Other names for putting together
10. Other names for shifts
11. Sutherland's classification
12. Kansky's models
13. Adding ratio comparisons
14. Other use meanings for addition
15. Redundant use classes for addition
16. Formulas involving addition

1. The general context. In what is a simplification of the steps in problem-solving described in Polya's How to Solve It (1957), many elementary school textbooks in the United States have adopted a multi-step guide to problem solving like the one that follows (Bolster et al., 1980).

Step 1:	read	[Polya	"Understanding the Problem"
2:	decide		"Devising a Plan"
3:	solve		"Carrying Out the Plan"
4:	answer		"Checking"]

The ideas in Part I are designed to help the student read and comprehend numerical information and thus could be said to attack the first step in problem-solving. Yet often the second step, deciding what to do, causes much more trouble. Even though many books have special pages devoted to the notion of "choosing an operation", the student is usually given little more than intuition to help in that choice. One of the benefits of having a rather complete set of use meanings is to give both teacher and student help in making such choices.

2. The classification process. We sorted in the following manner. First we chose an operation, say subtraction. Next we collected many problems of situations in which two numbers a and b were given and a-b was the answer. (For example,, I have $20. You have $25. How much more do you have than I?) The collection of problems gave rise to the use classes, into which we sorted the problems. Some of the use classes were more fundamental than others, in that they seemed to give meaning to the

BEST COPY AVAILABLE

operation. Those more fundamental use classes became what we call use meanings in this book. (For example, the subtraction problem just given illustates the comparison use meaning of subtraction.)

3. Classification criteria. We categorized examples of a given operation by the roles the numbers play in the situation. We illustrate with two uses of subtraction that involve the same numbers in different roles.

> A person gives away 3/4 of his or her estate.
> How much is left? [Classify as take-away.]
>
> Two boards are 3/4" and 1" thick, respectively.
> How much thicker is the second board? [comparison]

The subtraction in the examples is the same: 1 - 3/4. But in the first example,the 1 and the 3/4 play different roles (1 is the whole, 3/4 is given away), whereas in the second example, the 1 and 3/4 play the same role (each is one of two numbers being compared). This gives a clue to a difference in structure, and so these uses are placed in different classes.

It is often possible to conceptualize a use in more than one way, resulting in a choice of use class. For instance, consider a third use of the subtraction 1 - 3/4.

> A blouse now sells for 3/4 of its original price.
> What part was taken away?

This problem could be classified as comparison (part remaining to the whole), but the words also accurately suggest a derivation from take-away.

4. Classification non-criteria. Problems are often sorted by others in ways that do not help someone determine how the operations are used. The following ways of sorting were not used.

We did not sort by the particular algorithms or other procedures through which sums, products, quotients, etc., might be obtained, even though that kind of sorting is very common in elementary school arithmetic. For example, in a situation requiring 30 to be divided by 25, we did not care whether the solver does the problem "in his head", uses long division, short division, repeated subtaction, successive approximations utilizing multiplication, or a calculator. We were concerned only with the role of divisor, dividend and quotient in the real situation.

We did not sort problems by the size of numbers used. Finding a distance given a time of 19 1/2 hours and (average) speed of 90 kilometers per hour falls into the same use class as finding a distance given and elapsed time of .033 seconds and a speed of 6 centimeters per second.

We regarded equivalent forms of numbers as completely interchangeable; for example, "25% discount" and "1/4 off" are identical in use, and in working with them one might use .25 instead of either 25% or 1/4. Although some particular situations employ numbers in certain specific forms (for example, stock market quotations are in fractions of dollars, not in decimals, and most scientific situations use decimals or scientific notation), these are instances within use classes, not classes in themselves. Similarly, equivalent quantities, such as 1 inch and 2.54 cm, were considered interchangeable.

We did not classify by the particular semantic clues found in word problems (e.g., "of", "difference", "more", etc.) that may suggest an arithmetic operation because such semantic clues are frequently misleading and because problems from the real world often do not possess reliable semantic guides.

5. Other kinds of meanings for operations. The notion of meaning in arithmetic was given its greatest impetus by Brownell (Weaver and Kilpatrick, 1972), who wrote extensively on the subject in the middle half of this century. Brownell often contrasted teaching for meaning with rote teaching, and so "knowing the meaning" of something is often associated with "understanding" that something. Since "understand" is a rather vague word, it does not help to clarify what is meant by "meaning", but it does place the issue in a context that is a little easier to deal with. We ask: What does it mean to understand an operation?

A first kind of understanding is based upon numerical results. Many would call this the lowest level of understanding; some would hardly call it understanding at all. In this notion of meaning, one knows what subtraction means if one knows the answers to subtraction problems 7 - 4, 33.2 - 2.5, etc. This is the notion of meaning upon which one common formal definition is based, namely that subtraction is an operation that associates a particular result (a - b) with two given numbers (a and b).

A second kind of meaning or understanding is based upon knowing the processes or algorithms by which answers are found. One knows this meaning of subtraction if he or she can "do subtraction". In this meaning, subtraction of whole numbers is a process whereby one sets up the numbers in a column and takes differences, borrows, etc. In this kind of meaning, subtraction of fractions has a different meaning from subtraction of whole numbers because it is done in a different way.

A third kind of meaning of an operation is based upon its mathematical properties. There are two such meanngs in common use for subtraction. One is that subtraction "undoes" addition. The undoing yields related facts. E.g., since 47 + 13 = 60, we may conclude that 60 - 13 = 47 and also that 60 - 47 = 13. In general, defined this way, a - b= c if and only if c + b = a. The other meaning via a mathematical property is that subtraction is the inverse (or opposite) of addition, or to say that subtraction and addition are inverse operations. This definition

BEST COPY AVAILABLE

is often employed when negative numbers are under discussion. Then 3.2 - 2.5 = 3.2 + -2.5, and in general, a - b = a + -b. One knows this type of meaning of subtraction if he or she can "apply the properties".

A fourth kind of meaning of an operation is based upon *representations* of that operation. The repesentations may be concrete, graphic, or symbolic. For example, we might say "With Cuisenaire rods, subtraction means to take two rods, place them side by side with one pair of ends aligned, then see what rod you need to put next to the shorter rod to make the other ends even". Or we might say "On the number line, subtraction means...".

A fifth kind of meaning, based upon *uses* of the operation, is covered in this volume. One knows the use meanings of subtraction if one can "apply subtraction in the real world".

The people who understand subtraction best know all of these meanings.

6. *The work of Sutherland*. After these materials had gone through several drafts, and after a first draft of these notes were written, we came across a book by Ethel Sutherland that represents an earlier (1947) attempt to categorize uses of operations. She states the problem as follows:

> "To determine how many different one-step patterns there are in connection with each of the four operations, addition, subtraction, multipliction, and division". (p. 5)

Sutherland examined "all the verbal problems in a group of modern basal textbooks in arithmetic covering the work of Grades 3 to 6 inclusive". She looked at multi-step problems as well as single-step problems and from this determined the frequency with which the one-step patterns occur in the problems at each grade level. Each of the four fundamental operations constitutes a chapter of her book, in the following order: subtraction, division, multiplication, addition. We compare her classifiction to ours in the notes following our corresponding chapters.

Sutherland's work has been lost to later students of uses and meanings of operations. Even Vest and Kansky (see note 7), with extensive reviews of the literature, missed her. We believe this is due to the fact that her dissertation (at Teachers College, Columbia University), upon which her book is based, is not abstracted in *Dissertation Abstracts*, and this book appears as part of a general education series not likely to be examined by mathematics educators. We happened upon her work through a reference in Wheat (1951, p. 337). Future students should be careful not to ignore her valuable work.

7. *The works of Vest and Kansky*. The reader interested in various kinds of meanings of operations will be well served

BEST COPY AVAILABLE

by consulting the doctoral dissertations of Vest (1968) and Kansky (1969). Each dissertation contains a thorough review of the literature and an attempt at cataloguing representations of operations of arithmetic.

Kansky identifies four types of number system models and allows that there may be others: abstract models, _ad hoc_ models, application models, and structural models. "The application model is a problem situation--real, projected, or fanciful--which requires the use of mathematics but does not necessarily generate the mathematical concepts needed". (p. 114) He argues rather strongly that application models are not appropriate for introducing students to elementary school arithmetic, a position with which we disagree. In fact, we feel that students learn addition and subtraction quite well principally because teachers exploit some application models (e.g., putting together 3 objects and 4 objects to illustrate 3 + 4) from the first time the operations are taught.

Vest's models include what is usually called concrete embodiments (e.g., Cuisenaire rods) or representations (e.g., the number line) of the operations. He gives criteria for judging such models for appropriateness in the classroom that could also be applied to judging the appropriateness of use classes. These criteria are: general growth in the mathematical domain (e.g., in axiomatics or problem solving); specific knowledge, skill, and concept associated with the system of whole numbers; extension beyond the system of whole numbers; application to other specific mathematical skills (e.g., the teaching of percentage); concordance with the nature of the learner; level of difficulty; appropriateness of cognitive structure and style; contribution to the application of abstract principles; facilitation of methods of instruction; applicability to science; applicability to common social usage; applicability to teachers, standardized tests, and equipment.

8. _Work of others_. We are aware of no others who have attempted to classify uses of _all_ the operations in a coordinated way. However, there are many who have classified use meanings and other meanings of single operations and, with the recent boom in research on problem solving, there are a number of people who have tried to classify word problems requiring a given operation. A few other classification schemes for individual operations are given in the notes following the corresponding chapters.

The preceding comments have dealt with issues encompssing all of Part II. The remaining comments refer specifically to uses of addition.

9. _Other names for putting together_. This use class is often called _union_, taken from its association with measure functions, e.g., functions possessing the following property: If m is a measure function, A and B are disjoint sets, m(A) = a, and

BEST COPY AVAILABLE

$m(B) = b$, then $m(A) \geq 0$ and $m(A \cup B) = a + b$. Examples of functions of this property are;

$m(A)$ = count of set A
$m(A)$ = measure of angle A
$m(A)$ = length of segment A
$m(A)$ = area of region A
$m(A)$ = probability of event A in a given sample space

In Usiskin (1976), the count and measure instances are split into two different categories, entitled *union* and *joining*. This is an apropriate split if one wishes to emphasize the non-counting uses of addition. Many others have considered this idea, but most restrict themselves to counting situations. Carpenter, Hiebert, and Moser (1981) differentiate between instances where the objects being counted are together, in which case they are collected under the *part-part-whole* category, and where one set of objects is brought to the other, in which case the category is entitled *joining*. Such detailed analyses are necessary if one wishes to understand why children can do certain problems in this use class but not others that seem to involve the same kinds of notions. Greeno (1978) calls these *combine* and *exchange-increase* (more like our shift); Lindvall (1981) calls them *putting sets together* and *getting more things*.

10. *Other names for shifts*. The Greeno (1978) *change-increase* and Lindvall (1981) *getting more things* can be interpreted as alternate names for shift. Usiskin (1976) uses *slide*. The word shift is preferred here for two reasons: first, shifts can be both smooth (continuous) and jumpy (discrete), whereas slides connote only smoothness; second, shifts are neutral with respect to direction, whereas the notion of increase or getting more things is one-way.

11. *Sutherland's classification*. Sutherland's examination of word problems in four series of Grade 3-6 textbooks in 1947 led to four patterns for addition, all falling under the rubric of wanting to find a total. (Of the four operations, her weakest classification is that of addition.) Here are the four patterns, the total number of problems involving these patterns in the four series, and the range of percentages of addition problems filling in each pattern for the series.

Pattern 1: The phraseology in which the problems are expressed helps to emphasize the idea of finding a sum.
Total 931: Range 29%-43%

Pattern 2: The phraseology in which the problems are expressed does not contain characteristic words or expressions such as those listed in Pattern 1.
Total 675: Range 25%-30%

Pattern 3: The phraseology of the problems is peculiar to

the activity of buying and selling.
Total 788: Range 26%-37%

Pattern 4: The phraseology of the problems is similar to that used in certain subtraction patterns.
Total 74: Range 1%-5%

The patterns do not correspond easily to the use classes in this chapter, for all are based upon semantic or contextual considerations. For instance, the fourth pattern, which seems to correspond to our addition from subtraction use class, also involves problems we would classify as shift. Notice how many problems involve money (Pattern 3) and how few are of Pattern 4. We wonder if the same distribution holds in today's books. If it does, it might explain why students can do certain types of problems and not others.

12. Kansky's models. Kansky (1969) found the following general types of models in use for addition of natural numbers.

counter models
segmented-rod models
non-segmented-rod models
geometric models
storyline models

The storyline models are closest to the considerations here. Kansky finds three of these storylines: trips on a road, cash delivery, and balance beam. Trips on a road are akin to a number line. Cash delivery consists of a mail carrier bringing bills or checks. (Giving mail back to the carrier can model subtraction.) The balance beam is more of a physical model than a use class.

13. Adding ratio comparisons. Examples 10 and 11 of the putting together use meaning of addition involve the addition of numbers that are ratio comparisons. These very natural situations, with percentages and probability, provide counterexamples to the dictum occasionally heard that "you can't add ratios like other numbers". In order for addition of ratios to be meaningful, the ratio being added must have the same referent. (In Example 10 the referent is the total number of possible moves of the die onto or pas the hotels; in Example 11 it is the population of Manitoba.)

A situation often given for analyzing addition of ratios is as follows: A batter gets 1 hit in 3 at-bats in one game and 2 hits in 4 at-bats in a second game. Then the batting average (a rate or ratio, depending upon one's semantics) is .333 the first game and .500 the second, and surely the total is not .833. In fact the total is 3 hits in 7 at-bats for a combined batting average of approximately .429. But what is being added to obtain the combined results are the ordered pairs (1,3) and (2,4) yielding in the usual way for adding such pairs (3,7). The ratios are calculated after the addition, not before.

Thus it is possible both to add ratios as ordered pairs and to add ratios when they are single number ratio comparisons. The important thing to remember is not to use one type of addition when the other is more appropriate.

14. *Other use meanings for addition*. We considered but rejected *shortcut counting* as a meaning for addition. This idea covers situations often encountered when one first learns addition.

> Each day there were U.S. hostages in Iran in 1979-81, some communities raised a new flag in their honor. If there were 100 flags flying in such a community on a particular day, how many flags would be flying one week later?

To answer this question,instead of counting up by ones from 100, one applies the shortcut of adding 7. Shortcut counting has a mathematical counterpart in the definition of addition of natural numbers using the successor notion first offered by Peano, in what are called today Peano postulates.

The relationship between addition and shortcut counting is analogous to the relationship between multiplication and shortcut addition, and between powering and shortcut multiplication. Thus there are rationales for having this use meaning. We decided to the contrary because the examples, such as the hostage question above, were weak and, indeed, non-trivial examples were hard to find. However, others, in analyzing addition, might wish to consider including this use meaning, particularly if the behavior of very young students with addition is under discussion.

15. *Redundant use classes for addition*. Since all subtractions have addition counterparts, every subtraction use class suggests an addition use class. Consider the following comparison situation.

> On a day in January, due to "lake effect", the temperature in Chicago by the lake was 6°F at the same time that the temperature at O'Hare (about ten miles west of the Lake) was -4°F. What was the difference between the temperatures?

Here is a corresponding addition situation.

> In the winter in Chicago, temperatures by the lake tend to be about 10°F warmer than temperatures at O'Hare Airport (the official recording) due to what is called "lake effect", the warming of air due to lake water being warmer than the surrounding air. If the temperature was -4°F at O'Hare on a day in January, what was the approximate temperature by the lake?

Structurally, this addition situation falls under the rubric *recovering second number in a comparison*. However, the situation is easily placed in shift (shift 10°F to get temperature by the lake from temperatures at O'Hare; shift −10°F if the converse situation is present).

A second example is simpler.

> John had 35¢. Sue had 15¢ more than John. How much did Sue have?

We also would classify this as a shift situation even though two different quantities are involved. However, if one wished a more detailed analysis of uses, we could envision a separate use class for situations of this type.

We found no instances that could not be treated in one of the existing use classes, so we considered _recovering second number in a comparison_ as redundant.

16. _Formulas involving addition_. We recognize that many formulas involve addition (e.g., $F = 9/5\ C + 32$). These uses are analogous to the derived formula constant uses of number (Chapter 1, Section F). We have not included these uses because (1) in principle, the underlying uses are categorizable elsewhere (see Example 3, Section 8, Chapter 10) and (2) the operation is explicitly exhibited, and so no choice of operation is required.

BEST COPY AVAILABLE

CHAPTER 6

USES OF SUBTRACTION

Subtraction, like addition, has two basic use meanings, take-away and comparison. There are two other use classes, both derived from addition, for a total of four use classes in this chapter.

A. Take-away

B. Comparison

C. Subtraction shift

D. Recovering addend

Subtraction Use Class A: Take-Away

Take-away is the use meaning of subtraction most often encountered in school work. Here is an example with numbers larger than is typical in books. If 60 seats will be covered by an extended stage in an auditorium that has 1200 seats in all, how many seats will remain? The answer is 1200 - 60 seats.

Take-away use meaning of subtraction
Given amount - amount taken away = amount remaining

In the real world, "taking away" often undoes "putting together." The correspondence mathematically is that if $c - a = b$ (that is, a is taken from c to yield b), then $a + b = c$ (that is, a can be put with b to yield c). Like its counterpart, take-away has a wider range of applicability than is usually seen in school work. Take-away can be meaningful when the original quantity is a count, measure, or ratio comparison. The examples involve each of these uses of numbers.

Examples:

1. Counts. A social club in Toledo had 24 members until two members moved to Cincinnati. How many were left?

   Answer: 24 members - 2 members = 22 members.

   Comment: With very young students, it is important to consider the possible verbs which can imply the notion of "take-away". Here the verb is "moved".

2. Money. Starting the day with a dollar, a person spent 20¢ for a paper. How much money remained?

   Answer: \$1.00 - \$.20 = \$.80, so 80¢ remained.

   Comment: The situations of Examples 1 and 2 are so well-known to students that they are useful for teaching subtraction. That is, instead of applying subtraction to find the answers to these questions, there may be circumstances in which it is wise to use the answers to questions like these to help students learn when to do subtraction.

3. Time. Under voluntary regulations, the major TV networks allow a maximum of 9 1/2 minutes each hour for advertising and station breaks. How much does that leave as a maximum for program length in a given hour?

   Answer: 1 hour - 9 1/2 minutes = 60 minutes - 9 1/2 minutes = 50 1/2 minutes

   Comment: There is no minimum that must be taken away except for station breaks. So a program could be very nearly 60 minutes long.

4. Length. A gift giver estimates that 13 ft of a 30 ft roll of wrapping paper has been used. About how much remains?

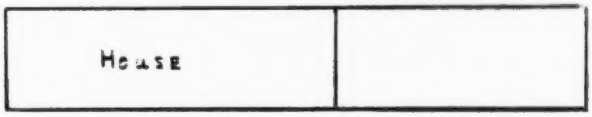
   Answer: 30 feet - 13 feet = 17 feet.

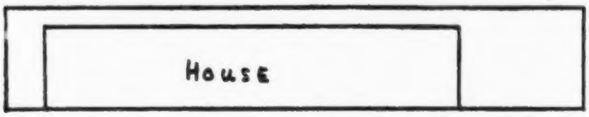
   Comment: Wrapping paper is measured by its length, not its area, though the latter more closely describes how much a person will need to cover a given surface.

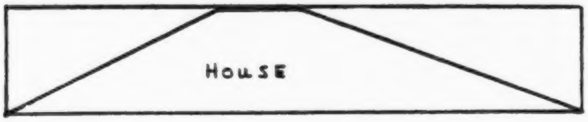
5. Area. A city building lot is 25' x 140' (hence 3500 sq ft), on which is to be built a house with 2000 square feet on the ground floor. How much land in the lot will remain for landscaping, access, etc.?

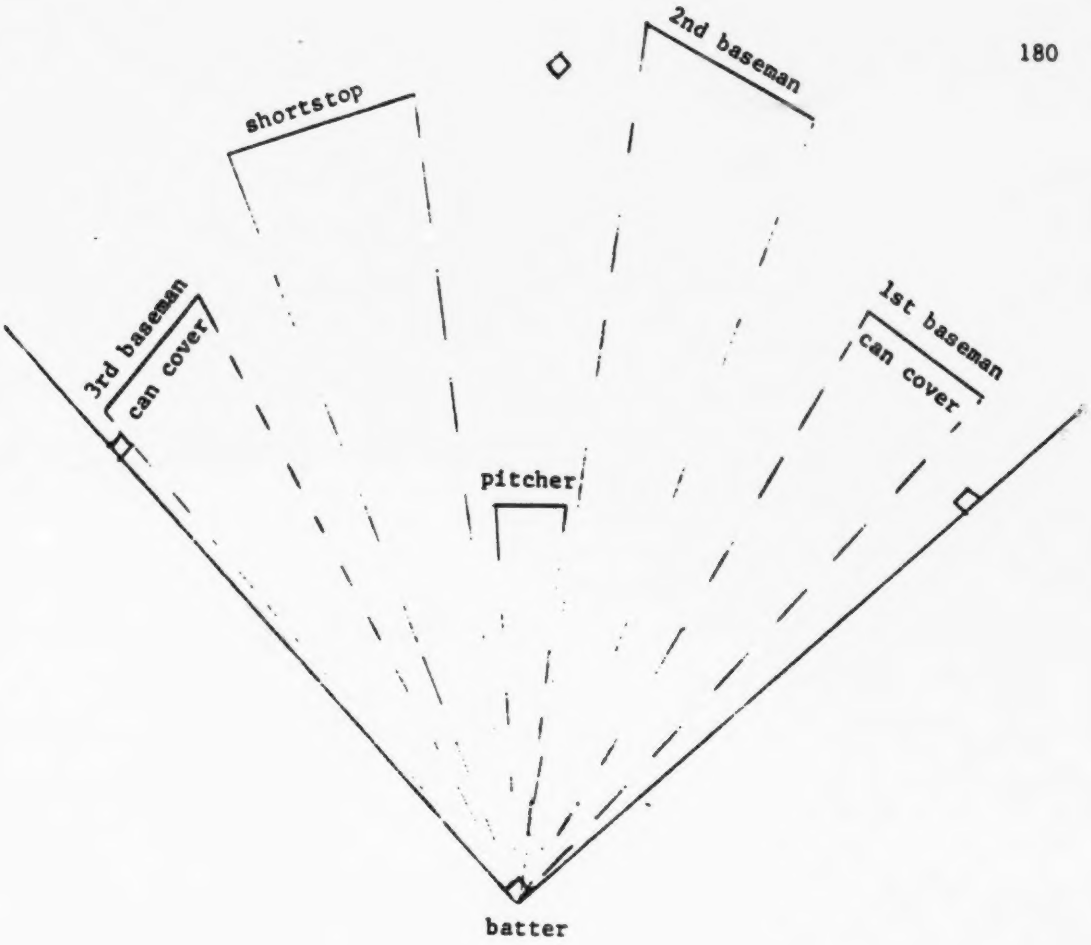
   Answer: 3500 sq ft - 2000 sq ft = 1500 sq ft, the area of land that will remain.

   Comment: The area that remains is independent of the shape and location of the house. Here are three possible layouts:








2nd baseman
shortstop
3rd baseman
can cover
1st baseman
can cover
pitcher
batter

6. Angle measure. In baseball, the batter hits into a playing field of 90° width between the foul lines. Suppose each of the four infielders can cover about 13° of angle on a ground ball hit hard, and the pitcher can cover about 6°. How much is left for the hitter to hit through?

Answer: 90° - 13° - 13° - 13° - 13° - 6°, or 32°. This would imply that about a third of ground balls hit hard will get through the infield.

Comment: Players try to position themselves so that there will be as little overlap as possible. The slower the ground ball, the larger the angle a player can cover. Medium speed ground balls are almost always reached.

7. Probabilities. If there is a 70% chance of rain tomorrow, what is the chance that it will not rain?

Answer: 100% - 70% = 30%.

Comment: 100% is the probability that there will be weather tomorrow.

8. Percents. From 1963 to 1972 in Central America the yearly birthrate was 4.4% and the death rate was 1.0%. Ignoring immigration and emigration, what was the annual rate of population increase?

Answer: 4.4% - 1.0% = 3.4%. (Population growth in the absence of migration is births minus deaths.)

Comment: This was the highest growth rate in any sector of the world at that time. Such a growth rate, if it continued, would result in a doubling of the population in about 21 years.

9. A small stadium has tickets in three price ranges: 2000 cheap, 2500 moderate, and 750 expensive. There are 247 cheap and 86 expensive tickets remaining. How many have been sold?

Answer: This situation is derived from take-away subtraction, but we wish to determine the amount "taken away".

For the cheap tickets: 2000 - (number sold) = 247,

from which number sold = 2000 - 247

= 1753.

In similar fashion, one can calculate that 2500 moderate and 664 expensive tickets have been sold.

Comment: This situation, like Example 1 of Section B, can be classified as "missing subtrahend" subtraction. In general, whenever a - b = c, then a - c = b.

Comment: When the given information is stored as the 3-tuples (2000, 2500, 750) and (247, 0, 86), the component by component subtraction exemplifies vector subtraction.

Subtraction Use Class B: Comparison

Mary is 13. Her sister is 8. How much older is Mary than her sister? This question, an easy one for children to answer, involves subtraction (13 years - 8 years = 5 years) but not in a take-away situation. Nothing is actually being taken away; the two ages are being compared. Comparison is a second fundamental use meaning of subtraction.

Comparison use meaning of subtraction

Given numbers or quantities a and b,

a - b tells by how much a and b differ.

If one can compare a to b, then one can compare b to a. The answers will be opposites, both literally and mathematically. For example, suppose you wish to compare the cost of an item with the amount you have available to spend.

Amount available		Cost of item		
\$100	-	\$97	= \$3	You have \$3 more than you need.
\$97	-	\$100	= -\$3	You need \$3.

Generally, we compare in the direction which gives a positive answer for the result. That is, we subtract the smaller number from the larger. But in situations where many comparisons have to be made, and when these comparisons must be uniform, the subtractions may lead to negative answers about as often as positive answers.

Comparison is meaningful with counts, measures, ratios, and with some locations. Comparison is generally not meaningful for codes.

Examples:

1. Measures. Bill weighs 40 kg and Cathy weighs 32.5 kg. How much more than Cathy does Bill weigh?

   Answer: 40 kg - 32.5 kg = 7.5 kg

   Comment: We would never ask "How much more than Bill does Cathy weigh?", but if the subtraction were done in the implied order the answer would be -7.5 kg. We would find that Cathy weighs "negative more", that is, less, than Bill.

2. Counts. Compare the numbers of students and faculty at the University of Georgia with the numbers of Georgia State University in Spring, 1978.

Georgia	21,665 students	1693 faculty
Georgia State	20,686 students	836 faculty

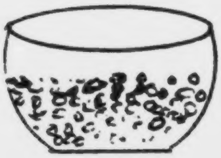
Possible answer: Georgia had 979 more students and 857 more faculty than Georgia State at that time.

Comment: We tend to use subtraction to compare when two numbers are relatively near each other, and ratio division to compare when one is at least double the other. For example, if a third university had only 2000 students, we might say that Georgia State had 10 times as many students (rather than 18,000 more), having calculated the 10 by dividing 2000 into 20,000. In the above question, we might say that Georgia had about 1000 more students and more than twice as many faculty members as Georgia State.

Comment: Georgia State, located in Atlanta, has many more part-time students than the University of Georgia, located in Athens. Since part-time students take relatively fewer courses, fewer teachers would be needed and this probably accounts for some of the difference in student-faculty ratios. But there may be other causes of these differences. Discussion of such causes can be one of the most valuable outcomes of teaching applications.

3. Error.

   John estimates that there are 325 marbles in the bowl. Maria estimates 500. The actual count is 422. Who is closer?



Answer: 500 - 422 = 78; Maria is 78 too high.

325 - 422 = -97; John is 97 too low.

Maria is closer.

Comment: Notice how subtracting in a consistent order shows not only how far the estimates are off, but also the direction of error.

Comment: A guess of 422, right "on the button," would yield the following subtraction:
422 - 422 = 0
That is what is meant by "zero" error.

4. Ratio comparisons. In 1980, the mortgage rate on newly purchased houses went from 12% to as much as 18%. How much of a change is that?

Answer: 6% increase in rate, and a considerable increase in resulting house payments.

Comment: The 6% difference between 18% and 12% represents a much greater difference in house payments than a 6% difference in smaller percentages, such as 11% and 5%. Thus subtraction alone should not be used to make decisions in this kind of situation.

5. Probabilities. In tossing two fair dice, the probability of a sum of 7 is $\frac{1}{6}$; the probability of a sum of 11 is $\frac{1}{18}$. How much more likely is a 7 than an 11?

Answer: $\frac{1}{6} - \frac{1}{18} = \frac{1}{9}$

Comment: By converting to decimals, we see that a 7 is about 11% more likely than an 11.

6. Large numbers. In March 1974, on the first "bargain-rate" Sunday, 558,860 persons rode the Chicago Transit Authority buses and trains. On the same Sunday a year before, 353,969 persons rode the CTA. How many more people rode the CTA on the 1974 Sunday?

Answer: 558,860 - 353,959 = 204,901

Comment: The difference could be used to determine how many more trains or personnel might be needed.

7. Rates. People in the United States eat an average of 3300 calories per day. People in Bolivia average 1870 calories per day. What is the difference in average caloric consumption in these two countries?

Answer: 1540 calories per day.

Comment: The difference seems quite great, and may be one cause of the 26 year difference in life expectancy (73 years in U.S. - 47 years in Bolivia.)

8. Locations. A person works from 6:15 a.m. to 11:00 a.m. with a 15 minute break. How many hours of work is this?

Answer: There is a 4 hr, 45 min interval between 6:15 and 11:00. We take 15 minutes off that, so the person worked for 4 hours, 30 minutes.

Comment: The two subtractions done to arrive at the answer are of two different use meanings, the first comparison, the second take-away.

Comment: The times 6:15 and 11:00 are locations in a reference frame, and subtraction comparison is meaningful. The actual computation is relatively difficult due to there being 60, not 100, minutes in an hour, so the numbers cannot be treated as if they were the decimals 6.15 and 11.00.

9. Temperatures. In a place with a low temperature of 20° and a high temperature of 32°, what was the daily range of temperatures?

Answer: The range, the difference between the high and low temperatures, is 12°.

Comment: The word "difference", often used to describe answers to subtraction comparison problems (and other subtraction problems as well) is one of those rare words that is appropriate in both technical and common usage and is not misleading in either.

10. Comparison from take-away. John bought a dozen eggs 3 days ago. Now only 5 are left. How many were used?

Answer: 1 dozen eggs - 5 eggs = 12 eggs - 5 eggs = 7 eggs.

Comment: Here we compared the original amount to what is left. In the next example, the comparison is not as obvious because the first number is so much larger than the second.

11. Missing subtrahend. Julie gave the cashier \$3.00 for lunch and received 60¢ in change. How much did lunch cost?

Answer: \$3.00 - \$.60 = \$2.40

Comment: Examples 10 and 11 could be classified as a missing subtrahend in a take-away subtraction. That is, in a take-away subtraction situation

$$\begin{pmatrix}\text{amount}\\ \text{to begin}\end{pmatrix} - \begin{pmatrix}\text{amount}\\ \text{used}\end{pmatrix} = \begin{pmatrix}\text{amount}\\ \text{left}\end{pmatrix},$$

we wish to find the amount used, the subtrahend. So we do the comparison subtraction

$$\begin{pmatrix}\text{amount}\\ \text{to begin}\end{pmatrix} - \begin{pmatrix}\text{amount}\\ \text{left}\end{pmatrix} = \begin{pmatrix}\text{amount}\\ \text{spent}\end{pmatrix}.$$

These equations are equivalent because $a - b = c$ exactly when $a - c = b$, the "double reverse" relationship between two subtractions.

12. Negative numbers. Michelle believes that the basketball team will win its next game by 12 points. By how much is her estimate off if the team (a) wins its next game by 5 points; (b) wins by 14 points; (c) loses by 11 points?

Answers: The error in her estimate may be found by subtraction. Using consistent order makes it easy to interpret the answers.

(a) $12 - 5 = 7$, an estimate 7 points too high,

(b) $12 - 14 = -2$, an estimate 2 points too low.

(c) $12 - (-11) = 23$, an estimate 23 points too high.

Comment: If the basketball teamwere predicted to lose (say by 6 points), the minuend in each of the problems would be -6, and if the team won the game by 5 points, the subtraction would be

$$-6 - 5 = -11,$$

correctly indicating that the estimate was 11 points too low.

Subtraction Use Class C: Subtraction Shift

In a trip from Los Angeles to New York, one changes time zones three times and must add three hours to keep up with correct time. Traveling back to Los Angeles, watches must be set back the three hours. We may consider these changes as adding 3 and -3 hours, respectively, or we may consider these changes as adding 3 and then subtracting 3. Though using only the operation of addition makes use of an elegant simplicity, most people prefer to use subtraction and avoid negative numbers. The subtraction illustrates a category of uses entitled subtraction shift.

$$\boxed{\text{Initial state - shift = final state}}$$

Examples:

1. Ages. An age guesser at a carnival gives a prize if your age is not guessed within 5 years. If you are 26 years old, how low can the age guesser guess and still not have to give a prize?

   Answer: 26 years - 5 years, or 21 years.

   Comment: The guesser can guess as high as 31 years and still not give a prize. The interval between 21 years and 31 years can be described as 26 ± 5 years, showing simultaneously both the addition and the subtraction shift.

2. Temperatures. In deserts, 30°C swings in temperature between day and night are not at all uncommon. If a daytime temperature is 35°C, what temperature should one prepare for at night?

   Answer: 35°C - 30°C, or 5°C, or not much above freezing.

   Comment: Whereas many subtraction shifts can also be classified as take-away, when scales are involved, as with temperature, the take-away conception is a little contrived.

3. Locations. In 1982, a woman died at the age of 64. When was she born?

   Answer: In 1918 or 1917, depending on her birth and death dates.

   Comment: Ages and years are time intervals, accounting for the lack of precision when they are subtracted.

4. Negatives. In Chicago, the place where the "official" temperature is read has been changed twice, first from downtown to Midway Airport, and then from Midway to O'Hare Field. In the winter, Midway is often 2° to 5°F colder than downtown, and O'Hare is often 3° to 5°F colder than Midway. If the record low for a January date is -13°F but was recorded downtown, what was a likely temperature then at O'Hare?

   Answer: From the given information, O'Hare is often 5° to 10°F colder than downtown.

   -13°F - 5°F = -18°F

   -13°F - 10°F = -23°F

   A likely temperature at O'Hare was -18° to -23°F.

   Comment: On January 10, 1982, a record low temperature of -26°F was measured at O'Hare. This roughly corresponds to earlier records at Midway and downtown.

5. Dow Jones Averages. The following paragraph appeared in the Chicago Sun-Times, October 28, 1982: "The Dow Jones average of 30 industrials, down 36.33 Monday and up 10.94 Tuesday, rose .28 to 1,006.35 by the close Wednesday." What was the price at the close Tuesday?

   Answer: 1,006.35 - .28 = 1,006.07

   Comment: This could be considered as a subtraction from addition (the next use class).
   Tuesday's close +.28 = 1,006.35

   Comment: Problems with extraneous data, like this one, are common in real situations but seldom found in schoolbooks.

Subtraction Use Class D: Recovering Addend

One of the authors weighed his 4-month-old baby on a doctor's scale by first standing on the scale without the baby, reading his weight as $154\frac{1}{2}$ lb. Then his wife gave him the baby, and together baby and father weighed $170\frac{1}{4}$ lb. How much did the baby weigh?

This problem is perceived by many people as connoting an addition situation.

$$\text{father's weight} + \text{baby's weight} = 170\tfrac{1}{4} \text{ lb.}$$
$$154\tfrac{1}{2} \text{ lb} \quad + \text{baby's weight} = 170\tfrac{1}{4} \text{ lb.}$$

However, one subtracts to get the answer, $15\frac{3}{4}$ lb. Of course the subtraction can be traced back to one or more use meanings, for example, a comparison of before and after. This use class consists of those instances in which one first thinks of an additive use but is forced to do subtraction. Mathematically, one begins with $a + b = c$ and utilizes either $a = c - b$ or $b = c - a$. Formulas involving addition often lead to uses in this class.

Examples:

1. Cost. After putting a \$20 bill in his wallet, George has \$63 there. How much did he have before putting in the bill?

   Answer: Since \$20 + previous amount = \$63, he had \$43.

   Comment: It is possible, but perhaps not as natural, to interpret this as an instance of take-away.

2. Distance. It's ten miles round trip from home to work for Ms. Robinson. If her automobile odometer shows that she traveled 25.3 miles in a day, how much of that driving was not to and from work?

Answer: 25.3 miles = total driven

= 10 miles + other driving.

So other driving = 25.3 - 10 or 15.3 miles.

Comment: This and Example 3 are particularly good instances of what some authors have called part-part-whole problems. A single part-part-whole problem is often considered an addition situation by some people and a subtraction problem by others. Experience seems to determine which is more natural.

3. Interest. A person opens a savings account and puts in $100/month.

After a year, there is $1200 plus interest, totalling $1239.72.

How much interest was earned?

Answer: Principal + interest = Total

1200 + interest = 1239.72

Subtracting, the interest is $39.72.

Comment: Most of us relate subtraction to addition so well that we set this problem up as subtraction from the beginning even though the situation is additive.

4. Formulas. Celsius and Kelvin temperatures are related by the formula

$$K = C + 273.16$$

A temperature of 150 kelvins (often called 150°K) in the lab is what temperature Celsius?

Answer: $150 = C + 273.16$

So $150 - 273.16 = C$

$-123.16 = C$, so the temperature is -123.16°C.

Comment: Another common solely additive formula is that for the perimeter of a triangle, $p = a + b + c$.

SUMMARY

The chart summarizes the four use classes of subtraction discussed in this chapter.

Use class	Origin	a	b	a - b
take-away	use meaning	original quantity	quantity taken away	quantity left
comparison	use meaning	one amount	second amount	difference
subtraction shift	derived from addition shift via a-b = a+-b	original quantity	shift amount	final quantity
recovering addend	derived from addition via related facts	sum	one part of sum	other part of sum

School books tend to overemphasize take-away at the expense of giving importance to the other use classes.

Pedagogical Remarks

Using key words. Some books have suggested that students be taught to make choices between operations in problems by relying on key words or other semantic clues. The idea is to then purposely write the problems with those words to reinforce the idea of looking for the words. It is good pedagogy to reinforce what one wishes to teach, but the real world does not work so simply. Word clues cannot be trusted. Consider the following example, utilizing a different tense of "take away".

> After her brother took away 4 pieces of candy,
> Susan had 40 pieces left. How many did she
> start with?

The answer is, of course, found by adding 40 and 4; the key word approach might mislead the student to subtract.

The point here is that words like "take-away" and "shift" suggest addition or subtraction but do not determine that these operations are being used. The teacher should not set down rules which may apply only to carefully constructed questions in the textbook in use.

Flexibility. The above guideline concerning the use of key words might be categorized under the heading "Be flexible". So, too, one should be flexible with respect to teaching the use classes. Students should see examples of all of the use classes for subtraction early (by the end of second grade), for if students see only take-away uses for five or six years, they will then find it difficult to recognize and cope with the vast number of comparison, subtraction shift, or recovering addend situations.

Flexibility also applies to use class interpretations of problems. The two authors often find themselves able to interpret the same problem in three or more ways.

Getting started. One way of familiarizing students with the use classes of subtraction is to pick a set of three numbers related by addition and subtraction, e.g., 85, 16 , 69, and have students make up realistic situations in which any two of the numbers are given and the third is the answer. For example,

> The temperature was 85°. It went down 16°. What was the new temperature?

or

> The temperature was 85° yesterday and is 69° today. By how much has the temperature changed?

or a slightly harder reworking of the data,

> The temperature, after going down 16°, is now 69°. What was it to begin with?

For situations like the last, it may be helpful to express the numbers in a subtraction equation $T - 16° = 69°$. and then use the related facts relation between addition and subtraction to solve the equation.

Subtraction of fractions and decimals. With use classes we make no distinction between subtraction of whole numbers, subtraction of fractions, subtraction of decimals, and subtraction of positive and negative numbers; the classes are independent of the numbers used. For example, one compares temperatures of -17° and 4° and lengths of $3\frac{3}{4}$" and $3\frac{17}{32}$ " by subtraction just as one would for whole numbers. Primary school teachers often do a fine job of giving students examples of all of the subtraction use classes, but unfortunately only for small whole numbers. Work with larger numbers is highly recommended. Middle school teachers should take some time from teaching algorithms and spend that time reinforcing with fractions and decimals the use classes students already know for whole numbers.

Questions

1-4. By omitting one of the pieces of the information given, convert each situation into three questions, two requiring subtractions and one requiring addition. Then identify the subtractions and the additions with one or more of their use classes.

1. Michael is now 9 years old. Two years from now he will be 11.
2. A 7.5 cm piece was cut off one end of a pipe of one meter length, resulting in a new pipe with a length of 92.5 cm.
3. Chicago has about 3 million people in the city and 4 million in its suburbs, for a grand total of 7 million in the Chicago metropolitan area.
4. The percentage of unemployed dropped .2% last month from 7.6% to 7.4%.
5. Ask three people to estimate the length of a table (in centimeters or inches). Measure the length. Calculate the error in the estimates so that negative numbers denote underestimates, positive numbers overestimates.
6. In 1978, the auto industry (U.S. and foreign) sold 15.4 million cars and trucks in the U.S. In 1981, 10.8 million cars and trucks were sold. Formulate the two subtraction problems suggested by this information.
7. Describe the additions and subtractions that need to be done to reconcile a bank statement with a checkbook record at the end of the month.
8. Refer to Section C, Question 5. What was the Dow Jones average at the close of the previous week?

9. What's the difference between "$\frac{1}{3}$ off" and "30% off"?

10. Invent two different subtraction situations that utilize fractions. Classify the situations you invent into the use classes of this chapter.

Notes and Commentary

1. Other classifications for uses of subtraction
2. Sutherland's classification
3. Variant names for use classes
4. Comparison as a use class name
5. Redundant use classes for subtraction
6. Considering addition and subtraction together
7. Vest's models for addition and subtraction
8. Use classes of subtraction and uses of numbers

1. Other classifications for uses of subtraction. For years, many methods books have given use classification schemes for subtraction, but often not for the other operations. For example, here is how Morton (1937) describes the "type problems of subtraction" (p. 188).

> "Subtraction is a process used to find the answers to three distinct types of problems. One involves the 'how-much-more' idea, another requires the finding of a difference, and the third is concerned with what is left".

Of these problem types, the first corresponds either to our recovering addend or comparison, the second corresponds to comparison, and the third corresponds to take-away. So Morton neglects only subtraction shift, but he would probably classify such instances under take-away.

Grossnickle and Reckzeh (1973) give the same three problem types as Morton, but use different names. Here are the names and the examples they give (p. 157).

> "Dick has 5 marbles but he lost 2 of them. How many did he have left? (Subtractive)"
>
> "Dick has 3 marbles but he needs 5 marbles. How many more marbles does he need? (Additive)"
>
> "Dick has 5 marbles and Tom has 3 marbles. How many more marbles does Dick have than Tom? (Comparison)"

We would call these take-away, comparison, and comparison. (We would also use girls' names as often as boys' names!)

Kennedy (1970) identifies four kinds of subtraction situations, two of which we would call comparison, one take-away, and one recovering addend. He gives a name only to the take-away type. Swenson (1973) names the three types of problems as take-away, comparison, and additive.

Carpenter, Hiebert, and Moser (1981) use part-part-whole, separating, and compare to identify their categories. Greeno

(1978) uses combine (even for subtraction situations), change-decrease, and compare. His use of change-decrease corresponds to our subtraction shift.

Usiskin (1976), uses take-away and cutting-off for the discrete and continuous instances of what we call here take-away and directed distance for what we call comparison.

2. Sutherland's classification. Sutherland (1947) gives four general categories, split into ten subcategories, as follows:

Patterns that involve the remainder or "how-much-left" idea:

1. A given amount has been decreased. Find out how much is left or remains.
2. A given amount of money has been decreased by a purchase. Find the amount of change received.
3. A given amount has been decreased. The amount left or the change received is known. Find the amount taken away, spent, sold, given away, etc.

Pattern that involves the "how much more" or building up idea:

4. A given amount is on hand and a given larger amount is desired. Find how much more is needed to equal the desired larger amount.

Patterns that involve the comparison or difference idea:

5. Two unequal amounts are given. They are to be compared by finding how much more or how much less one is than the other.
6. Two unequal amounts are given. They are to be compared by finding how much larger, longer, taller, older, etc., one is than the other.
7. Two unequal amounts are given. They are to be compared by finding their difference.
8. Two unequal amounts are given. They are to be compared by finding their difference but the word difference is not used in the problem.
9. Two unequal amounts have been compared. One of the amounts and the difference between them are given. Find the other amount.

Pattern that involves the "separation-into-parts" idea;

10. Given the sum and one part, find the other part.

Patterns 1-2 are take-away, 4-8 are comparison. Patterns 3 and 9 cover what we called a "missing subtrahend" type of comparison (Examples 11 and 12 of Section B). Pattern 10 corresponds to our recovering addend. Generally, except for these patterns, we see that books covered all of Sutherland's patterns, and an analysis of her problem examples shows that these cover all of the use classes in this book, though we would not separate the patterns into groups as she did.

Sutherland's analysis of all word problems in four grade 3-6 series revealed the following counts for the appearances of these ten patterns:

Pattern	Series X	Series Y	Series Z	Series W
1	185	182	198	225
2	66	67	51	92
3	17	13	41	24
4	71	56	84	108
5	139	218	83	91
6	64	80	41	59
7	89	13	13	23
8	87	102	138	72
9	3	2	5	5
10	53	63	30	45

3. Variant names for use classes. The above discussion suggests alternate names for some of the use classes. Closer variants would be "taking away", grammatically consistent with the corresponding "putting together" use meaning of addition, and "missing addend", a common phrase (e.g., see the lengthy discussion in May (1974)), in place of "recovering addend".

We feel that there is something missing in every problem; thus "missing addend" incorrectly infers a special type.

We decided not to use the name "taking away" because the more direct "take-away" is so etched in the minds of teachers and students. To be consistent with grammatical forms would entail many changes in language for multiplication and division as well.

4. Comparison as a use class name. Numbers may be compared either by subtraction or division. For example, the population of the U.S. was 50 million in 1880 and 230 million in 1980. We could say:

> The 1980 population is 180 million more than the 1880 population.
> (Comparison by subtraction: 230 - 50 = 180)
>
> The 1980 population is about 4.6 times the 1880 population.
> (Comparison by division 230/50 = 4.6)

Division comparison we call ratio. The names comparison (for comparison by subtraction) and ratio (for comparison by division) are so common in the literature that we did not feel we should change them.

5. Redundant use classes for subtraction. For two reasons subtraction shift is almost a redundant use class. First, if negative numbers are allowed, addition shift can be thought to include it. Yet the subtraction action is more natural to many people than thinking of addition of negative numbers. Second, many people think of shift situations as take-away. However, there are subtraction shift situations that do not fit the take-away mold.

We considered a use class entitled recovering take-away or recovering subtrahend, as illustrated by Examples 10 and 11 under comparison, but all examples that we found and devised seemed almost as natural as instances of comparison.

6. Considering addition and subtraction together. There are many reasons for considering use meanings and classes for these two operations together. Carpenter, Hiebert and Moser (1981) and others have done that with the part-part-whole use class. Surely addition and subtraction shift belong together and a case can be made for considering comparison with either of the addition use meanings. An advantage of considering the operations together is that there is no need for the classes addition from subtraction and recovering addend.

Our reasons for separating addition and subtraction are pedagogic and editorial. In school texts and methods books, these operations are separate and for reference it seemed best to separate them. Also, if we combined addition and subtraction, for consistency, we should then have had to combine multiplication and division. Those operations are quite a bit more complex than addition and subtraction and we felt that the results would have been too involved. We also felt that it would be easier for others to combine things we had separated than to separate things we had combined.

From the standpoint of use meanings, however, there is a deeper reason for separating addition and subtraction (and later separating multiplication and division). There are many structural and pedagogical links among operations, and some of these carry over to closely linked use meanings, as with addition put-together and subtraction take-away or, even more obviously, with addition and subtraction shifts. Legitimate links of all sorts should be exploited in teaching the uses of arithmetic. But it is also the case that each operation has distinctive use meanings of its own not easily encompassed by the other operations. The distinctive use meaning for subtraction is comparison. In comparison situations there is often no part-whole nor is anything put together or taken away or shifted; there are merely two quantities present that remain unchanged, with the question of how much more or less one is than the other.

Such problems can of course be stated additively, e.g., "How much must John grow in order to be as tall as his mother?", but subtraction still is the operation relating the given numbers a and b to the answer, and the conversion to addition loses something in the process. (The comparable case for division is its distinctive uses in expressing ratios and rates, with no other operation serving those purposes in a direct way.)

7. Vest's models for addition and subtraction. Vest (1968) catalogs thirteen different families of models for addition and subtraction, treating the operations simultaneously because any model for one can be switched into a model for the other via related facts. His thirteen families are: set union, decomposition, comparison, machine-type, rod, number line, structured pattern, counting, scaler [sic] parts of vectors, operators, McLellan-Dewey, Minnemath, inverse. We recognize the first three of these as obvious counterparts to use classes, the machine-type as being close to shifts, the rod and number line as manipulative aids, the structured pattern, vector, operator, Minnemath, and inverse as mathematical (as opposed to applied) approaches, and the counting and McLellan-Dewey as special cases of others. Thus Vest's models attempt to include not only uses, but also physical and mathematical representations of the operations.

8. Use classes of subtration and uses of numbers.
Any of the use classes of subtraction (and addition as well) can involve numbers which are themselves used as counts, measures, or ratio comparisons.

When subtraction is done with locations, the situation is almost always a shift or a subtraction comparison. That is, one seldom puts together or takes away locations. For example, a 3rd place finish is 2 positions ahead of 5th place finish (subtraction in 5 - 3 = 2 is comparison); it does not make sense to take two places away from a 5th place finish. Codes are seldom subtracted (or added).

Derived formula uses lead to recovering addend.

CHAPTER 7

USES OF MULTIPLICATION

The uses of multiplication fall into three use classes of broad applicability, derived from two use meanings and the relationship multiplication has with division.

A. Size change

B. Acting across

C. Rate factor

Most of these uses are touched upon only briefly in children's school experiences with multiplication, perhaps because multiplication is explained not by its external uses, but as repeated addition. To us repeated addition is an algorithm applicable only to multiplication problems where one factor is a small whole number. With such a limitation, repeated addition does not and cannot cover any complete class of uses of multiplication. This may account for many of the difficulties that children have in applying multiplication.

Multiplication Use Class A: Size Change

The size change use meaning of multiplication involves a quantity and a factor which affects the size of that quantity. For example, we may begin with an item that cost \$50. What are some things that can happen to the \$50 quantity?

It could be tripled.	3 x \$50 = \$150, the new price.
It could be halved.	$\frac{1}{2}$ x \$50 = \$ 25, the new price.
A 4% sales tax	.04 x \$50 = \$ 2, the tax.
could be charged.	1.04 x \$50 = \$ 52, the total to pay.
There could be a	30% x \$50 = \$ 15, the savings.
"30% off" sale.	70% x \$50 = \$ 35, the sale price.

The factors 3, $\frac{1}{2}$, .04, 1.04, 30% and 70% are scalars, numbers without units, ratio comparisons in the context of uses of numbers. They arise from the desire to change the price by an amount related to the size of the price. For example, a discount or tax can be applied to everything in a store with bigger changes for higher priced items than for lower priced items. The general pattern of these examples constitutes a basic use meaning of multiplication.

Size change use meaning of multiplication

size change factor x original quantity = final quantity

In a size change use of multiplication, both the original and final quantities have the same unit. Above, the unit was dollars. The value of the size change factor signifies the particular type of size change to be effected, as shown in the following chart.

Values of size change factor	Types of applications
greater than 1	enlargement, amounts after interest, "times as many"
1	no change in size
less than 1	contractions, amounts after discount, "parts of"
0	annihilation
less than 0	directional change in conjunction with one of the above

There are a very large number of size change situations, and the verbal cues are quite diverse. For instance, the examples include such cues as time-and-a-half (multiply by 1.5), quadrupled (multiply by 4), 20% off (multiply by .20), 250 times, and 1/16 the size of. When a quantity is size changed twice, there are two multiplications to be performed. (You may want to read Example 12 below at this time.) The form is:

$$\left(\begin{array}{c}\text{second}\\ \text{size change}\\ \text{factor}\end{array}\right) \times \left(\begin{array}{c}\text{first}\\ \text{size change}\\ \text{factor}\end{array}\right) \times \left(\begin{array}{c}\text{original}\\ \text{quantity}\end{array}\right) = \left(\begin{array}{c}\text{final}\\ \text{quantity}\end{array}\right)$$

Due to the associative property, either multiplication can be done first, If the multiplications are done left to right, two size change factors will be multiplied. Obviously this can be extended to more than two factors.

Examples:

1. Suppose a job pays time-and-a-half for overtime. At \$4.25 per hour, how much is paid per hour of overtime?

   Answer: $1\frac{1}{2}$ x \$4.25, or \$6.375. (Employers might round down to \$6.37.)

   Comment: The size change factor here is $1\frac{1}{2}$. For double time, the size change factor would be 2.

2. From the years 1940 through 1980, consumer prices approximately quadrupled. Using this as a guide, what price in 1980 would compare to the price of a house which cost $30,000 in 1940?

   Answer: "Quadrupled" signifies a size change factor of 4.

   4 X $30,000 = $120,000.

   Comment: Home prices vary greatly depending on location and quality, so overall changes in consumer prices may not reflect values of individual houses.

3. A microscope lens magnifies 250 times. Viewed under this lens, a human hair .1 mm in thickness would appear to be how thick?

   Answer: 250 X .1 mm = 25 mm

   Comment: The word power often signifies the value of a size change. Thus binoculars which are 8 power have the effects of multiplying the apparent lengths in the object being looked at by 8. This is why "8X" is the symbol for "8 power".

4. Doll house furniture is often 1/12 the size of normal furniture. A typical normal chair can be about 45 cm high. How high would the corresponding doll house furniture be?

   Answer: $\frac{1}{12} \times 45 \text{ cm} = 3.75 \text{ cm}.$

   Comment: 1/12 is the size change factor. The word scale often signifies the value of a size change factor. It can be said that doll houses are "1/12 scale". The factor 1/12 is popular because it simplifies changing feet to inches. For the metric system, this factor is not as convenient.

   Comment: The scale factor 1/12 applies only to linear dimensions. With this scale, areas are multiplied by $(\frac{1}{12})^2$ or $\frac{1}{144}$, volumes by $(\frac{1}{12})^3$ or $\frac{1}{1728}$. A more extended discussion is found in Section A of Chapter 9, Powering.

5. A scale drawing of a house is to have the scale 1/4":1'. If the original house is 50' long and 30' wide, what will be the dimensions of the scale drawing?

Answer: 1/4":1' = 1":48" = 1:48. Multiply the dimensions by 1/48 to get 12.5" by 7.5".

Comment: One often sees the scale represented as 1/4"=1', which is shorthand for 1/4" on the drawing corresponds to 1' in the actual house. This is a use of the equal sign that does not connote equality, but in context causes little confusion.

Comment: In geometric terms, the scale drawing here is similar to the original and is a contraction of the original. The ratio of similitude of the drawing to the original is 1/48.

6. A $70 item is marked "20% off". How much discount is being offered?

Answer: 20% × $70, or $14.

Comment: The 20% is the size change factor.

Comment: In tryouts of these materials, at least one teacher was bothered by the multiplication of $70 by 20%. The teacher wanted $70 × .20. We emphasize that 20% is as much a number as .20. Though for the standard paper and pencil algorithm, one would convert 20% to .20 and multiply;

$$\begin{array}{r} 70 \\ \times\ \ .20 \\ \hline 14.00 \end{array}$$

conversion is not necessary. One could instead use a calculator with a percent key and press the following keys:

$\underline{7}\ \underline{0}\ \underline{\times}\ \underline{2}\ \underline{0}\ \underline{\%}$, which yields 14.

Someone else might multiply by $\frac{1}{5}$. The general point is that one should strive to be flexible in notation.

Comment: Percent notation, as found in Examples 6 - 9, is almost always found when a scalar is used as a ratio comparison. Thus a percent is never attached to a unit. For example, one never sees "30% miles" even though that is mathematically equal to .3 miles. One never sees "200% dollars" for 2 dollars. When a percent is used in multiplication, it is always either as a size change factor or derived from a ratio use of division.

7. A $70 item is marked 20% off. How much will you have to pay for the item?

   Answer: $56.

   Comment: It is typical to multiply the $70 by 20%, getting $14 as in Example 5, then to subtract $14 from $70 to get $56. It is more efficient (and better for some more complicated situations) to consider a 20% discount as signifying that you pay the other 80%. Then the answer can be calculated directly: 80% X $70 = $56.

8. A bank gives 6% yearly interest on special accounts. How much will $1000 grow to in a year?

   Answer: $1060.

   Comment: Instead of calculating the interest, it is easiest to think of the principal and 6% interest together as representing a scale factor of 1.06. Then

   1.06 X $1000 = $1060.

   This method is of particular advantage when considering interest on the interest.

   Comment: The same multiplication can be done to quickly obtain the total price with a 6% sales tax.

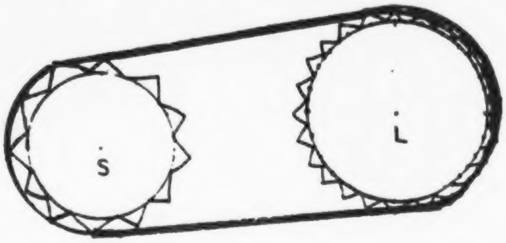
9. Expected value. In the United States, about 4% of people have blood type AB. About how many people with type AB should be expected in Green Bay, Wisconsin, a city of about 80,000 people?

   Answer: 4% x 80,000 people = 3200 people

   Comment: See Example 12, below, for an extension of this example.

10. In the picture below, the smaller wheel S has 12 sprockets, the larger wheel L has 24. So S will make two revolutions for every revolution of L. Thus S goes around twice as fast as L. If the larger wheel rotates at 250 rpm (revolutions per minute) what is the rotation rate of the smaller wheel?

Answer: 2×250 rpm = 500 rpm



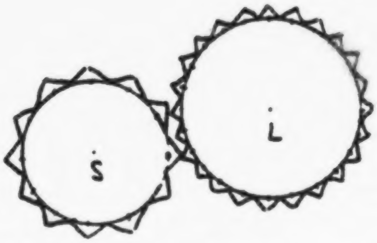
Comment: The 2 is the size change factor. One could also say that L rotates half as fast as S. Then the size change factor is $\frac{1}{2}$.

$$250 \text{ rpm} = \frac{1}{2} \times 500 \text{ rpm}.$$

Comment: If wheel A rotates $\frac{2}{3}$ as fast as wheel B, wheel B then rotates $\frac{3}{2}$ times as fast as wheel A. The relationship is a nice application of reciprocals.

11. If the wheels of Example 10 are moved to fit together and the belt removed, then when S goes around in one direction, L will go around in the opposite direction. One wheel can always be considered to be rotating clockwise, the other counterclockwise, which of these depending whether the observer is viewing from the front or back. To take the directions of the wheels into account, negative numbers can be used. Here S rotates -2 times as fast as L, that is, twice as

fast in the opposite direction.



Suppose we consider revolutions clockwise as positive, counter clockwise as negative. If the larger wheel turns -3 revolutions (meaning 3 revolutions counter clockwise) how many revolutions and in what direction will the smaller wheel revolve?

Answer: -2 x -3 revolutions = 6 revolutions. The 6 being positive indicates that the revolutions will be clockwise.

Comment: In general, for this situation:

$$-2 \times \begin{pmatrix}\text{number and direction}\\ \text{of revolutions of } L\end{pmatrix} = \begin{pmatrix}\text{number and direction}\\ \text{of revolutions of } S\end{pmatrix}$$

Comment: In the mathematical discussions of Rubiks's cube, a clockwise turn is usually taken as positive. In trigonometry and most classical mathematics, on the other hand, counterclockwise rotations are considered positive.

12. (Extending Example 9) Recall that about 4% of people in the U.S. are of blood type AB. Suppose that, on the average, about 60% of the population is old enough and well enough to give blood. In an emergency how many people in Green Bay (population about 80,000) might be available to give type AB blood?

Answer: 60% x 4% x 80,000 people = 1920 people.

Comment: Given that the original population and the percentages are estimates, and that what applies to the U.S.A. might not apply in Green Bay because of ethnic dissimilarities, the answer should be estimated as "about 2000 people".

Comment: The two size change factors are 60% and 4%. Since

$$60\% \times 4\% = 2.4\%,$$

one might estimate that about 2% of the population can give type AB blood.

13. Multiple discounts. In business, it is not uncommon to give wholesalers discounts on discounts. For example, a discount encountered in plumbing is "6 10's and a 20", by which is meant 6 consecutive discounts of 10% followed by a 20% discount. What one discount does that equal?

Answer: A 10% discount signals to multiply by 90% or .90, a 20% discount signals a size change factor of .80. The result is .90 x .90 x .90 x .90 x .90 x .90 x .80 x original ≈ .425 x original.

The amounts to a discount of between 57% and 58%.

Comment: It would be most cumbersome to calculate each 10% or 20% discount and subtract again and again.

Comment: Multiple discounts are not uncommon in wholesale plumbing. This problem was given to one of the authors by a plumbing sales representative who needed but did not know how to calculate the answer.

14. (Extending Example 1). Suppose 20% of your pay goes for taxes. If you make $4.25 per hour, and get time-and-a-half for overtime, how much per overtime hour are you making after taxes?

Answer: You get 80% if 20% is going for taxes.

$80\% \times 1\frac{1}{2} \times \$4.25 = \$5.10$, so you would have $5.10 per hour after taxes.

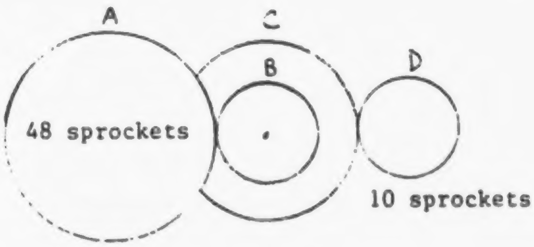
Comment: It usually pays well for hourly employees to work overtime.

15. (Like Example 3) A photographic enlarger is applied twice, once to magnify a picture 2 times, once to magnify it 3 times. What is the result?

Answer: 3 x 2 x (original lengths) = 6 X (original lengths), so the result is to magnify lengths 6 times (and areas 6^2 or 36 times).

Comment: This may seem very simple, but many people answer 5 times, adding instead of multiplying.

16. Compound gears. How many times faster, and in what direction, does D move relative to A?



outer C: 60 sprockets
outer B: 12 sprockets

Answer: revs of B = -4 revs of A, using the idea of Example 10.

revs of C = revs of B

revs of D = -6 x revs of C

= -6 x revs of B

= -6 x -4 revs of A.

= 24 x revs of A

Comment: Timepieces that run with gears work using these principles.

17. Probabilities of independent events. On the TV show "Let's Make a Deal", very popular in the 1970s, there were 3 doors with prizes, one of which concealed quite valuable prizes. What was the probability that a contestant would choose a valuable door twice in a row?

Answer: When two events A and B are independent (i.e., if the occurrence of one does not affect the occurrence of the other), then

$$\begin{pmatrix}\text{probability that}\\ \text{both A and B occur}\end{pmatrix} = \text{prob. of A} \times \text{prob. of B}$$

$$\text{and, in this case,} \quad = \frac{1}{3} \times \frac{1}{3}$$

$$= \frac{1}{9}$$

Comment: Producers might wonder if there were 100 contestants with the opportunity to guess twice, how many would win two valuable prizes? The most likely number of times this would occur (called the expected value in statistics) is

$$\text{prob. of A} \times \text{prob. of B} \times \text{no. of opportunities}$$

$$= 1/3 \times 1/3 \times 100$$

$$= 11\tfrac{1}{9}$$

or about 11 times. The two probabilities act as size change factors on the number of opportunities.

Multiplication Use Class B: Acting Across

If 3 editors work for 5 months to get a manuscript ready for publication, we say that the editing has taken 15 editor-months. If a 100-watt bulb is on for 14 hours, it uses 100 × 14 or 1400 watt-hours of energy. These situations exemplify a use meaning of multiplication we call acting across or acting through. The 3 editors act across a time interval of 5 months, and the 100 watts act through 14 hours.

In a rectangle, the length acts across the width to produce an area, the product of the two. Thus area is a special kind of acting across.

Area of shelf

$= 1\frac{1}{2}'' \times 10\frac{5}{8}''$

$10\frac{5}{8}''$ $\quad = \frac{3}{2}'' \times \frac{85}{8}''$

$= \frac{245}{16}$ sq in.

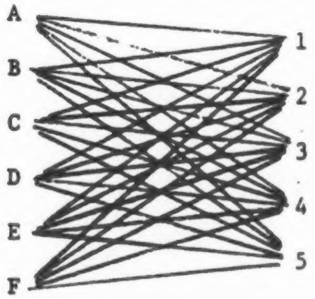
$1\frac{1}{2}''$ $\quad \approx 15$ sq in.

Area of tile

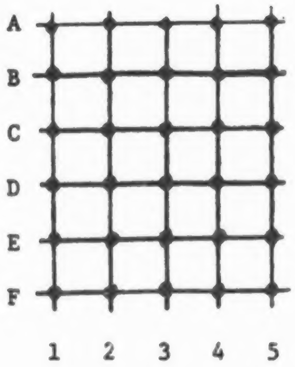
$= 25$ cm x 25 cm

$= 625\ \text{cm}^2$

Another special kind of acting across occurs in counting problems where one is interested in all possible pairs of objects from one set with those from another. For instance, most race tracks have a "daily double". A bettor must pick the winners of two races to win the daily double. Suppose six horses A, B, D, C, E, F run in the first race, and 5 horses 1, 2, 3, 4, 5 run in the other. The diagrams below pair each horse of the first race with all horses of the second race. The number of pairs of horses A-1, A-2, A-3, . . . F-5, is 30. One of these pairs is the winning daily double combination, so by just guessing, the chances of winning this daily double would be 1 in 30.



6 points x 5 points =
30 connecting segments



6 horizontal lines
x 5 vertical lines
= 30 points of intersection

Instances of acting across multiplication involve two quantities as factors. Each of one quantity acts across, or combines with, all of the other quantity. The resulting product quantity has a different unit than either factor, a compound unit. For example, where watts measure work and hours measure time, watt-hours is a unit of energy. The area of a rectangle is measured in square units, its sides in units of length. Unit analysis is discussed in more detail in note 6.

In geometry, the formula for the area of a rectangle is the one from which all other area formulas are derived. In combinatorics (the branch of mathematics dealing with counting problems), the number of pairs of elements in two sets, as calculated above for the special case of the daily double, is called a fundamental counting principle. Physics abounds with examples of acting across, force acting across distance, current acting across resistance, current acting across electrical pressure, etc. So this use meaning of multiplication is basic in a variety of areas.

Three factors can act across each other, and so this use class can involve products of more than two numbers. Volume in geometry and permutation problems in probability are common examples.

Examples:

1. What is the area of a rectangular plot of land that is 74' by 120'?

   Answer: The area of the plot is 74' x 120', or 8880 sq ft.

   Comment: An acre is 43,560 sq ft. Since $\frac{8880}{43560} \approx \frac{1}{5}$, this plot is about $\frac{1}{5}$ of an acre.

   Comment: Both "sq ft" and "ft^2" are abbreviations for "square feet". Students should be familiar with both.

2. How many square meters are in the floor of a 3.5 m by 4.5 m rectangular-shaped room?

   Answer: $3.5\ m \times 4.5\ m = 15.75$ sq m $= 15.75\ m^2$.

   Comment: The abbreviations m^2 for square meter, ft^2 for square foot, etc., are preferred by many because there is manipulation of units much like that of variables in algebra.

   $4\ m^2 + 3\ m^2 = 7\ m^2$ adding areas

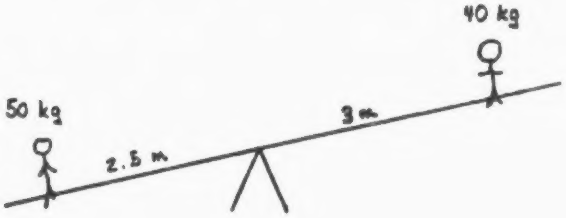
   $4\ m \times 3\ m = 12\ m^2$ to obtain area

   See note 7.

3. Product moment. A seesaw is an example of a lever, where the board rests on the fulcrum. The moment of a force is the product of the amount of force applied and distance from the force to the fulcrum.

   That is, Moment = Force x distance

   $$M = F \times d$$



Moments on both sides of the fulcrum must be equal for the seesaw to balance. Will a 50-kg person sitting 2.5 m away from the center of a seesaw balance a 40-kg person sitting 3 m away?

Answer: On one side here there is a moment of 50-kg x 2.5 m or 125 kg-m on the other side 40-kg x 3 m or 120 kg-m. The seesaw will almost balance; the 50-kg person should move slightly closer to the center (or the 40-kg person should move slightly farther away).

Comment: The unit for moments is kg-m, which suggests force acting over a distance.

4. Power. Power P (measured in watts), electrical pressure V (in volts), and current I (in amperes) are related so that

$$\text{Power} = \text{Electrical pressure} \times \text{Current}$$

That is, $P = V \times I$

One General Electric toaster oven is designed for a 120-volt circuit (the standard circuit in the U.S.) and uses 1350 watts of power. Will this toaster oven by itself blow out a 15-amp fuse?

Answer: $P = VI = 120 \text{ volts} \times 15 \text{ amps} = 1800 \text{ watts}$, so a 15-amp fuse on a 120-volt curcuit will handle 1800 watts of power, more than enough for this toaster oven.

Comment: Such an oven on the same circuit as an iron using 1000 watts would blow out the fuse (or trip the circuit-breaker).

Comment: To find I, the current required by the toaster oven, substitute:

$$1350 \text{ watts} = 120 \text{ volts} \times I$$

This equation can be solved by I by dividing both sides by 120 volts and shows that 11.25 amps are flowing through the circuit used by the toaster-oven.

5. Combinations. The school lunch counter will make up sandwiches in two kinds of bread (white and whole wheat) and three fillings (ham, cheese, or tuna salad). On how many days can one get a different kind of sandwich (one kind of bread, one kind of filling)?

Answer: There are 2 kinds of bread and 3 filling with each, so there are 2 x 3 = 6 kinds of sandwiches.

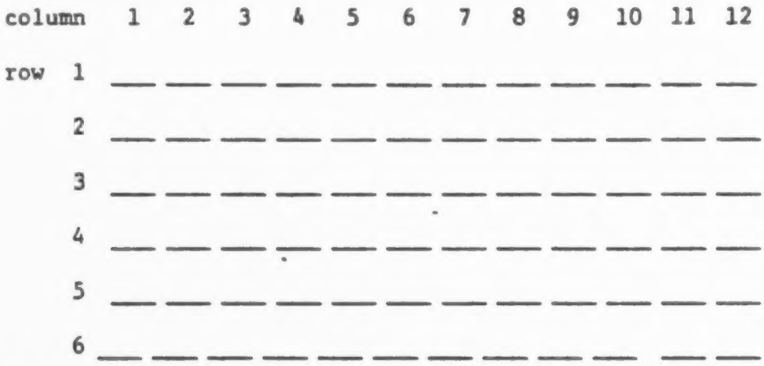
Comment: With counting units written, the multiplication is
2 breads x 3 fillings = 6 bread-filling combinations.

6. Couples. Seven girls and six boys try out for the two lead parts in a play. If the director of the play wished to try each girl with each boy, how many couples would the director have to try?

Answer: 7 girls x 6 boys = 42 couples.

Comment: If each tryout takes only 3 minutes, it would take about 124 minutes to observe all couples. (The multiplication of 3 minutes per couple times 42 couples is an example of the rate factor use class, discussed in Section C.)

7. Arrays. A small theater has 6 rows and 12 seats in each row. How many seats are there in all?



Answer: Think of 6 rows and 12 columns of seats. Each seat in the theatre is identified by the intersection of a row and a column. There are 6 x 12 or 72 such row-column intersections, so 72 seats.

Comment: This situation may be interpreted as rate factor multiplication. See Example 6, Section C. It also can be interpreted as area, by thinking of each seat as occupying a part of a 6 x 12 rectangle.

8. Factorials. In an attempt to make an election as fair as possible, an election board considers printing ballots with the names of 4 candidates for a position juggled in all possible orders. How many different ballots are needed?

Answer: Think of creating 4-tuples. The first component can be any one of the 4 candidates. Then the second component can be any of the 3 remaining, giving 12 possibilities for the first two. The third component can be any of the 2 remaining, for 24 possibilities. Once this is done, the fourth component is determined. Thus the answer is 24, or 4 × 3 × 2 × 1.

Comment: The number of ballots is more than one might at first think, and accounts for this idea seldom (if ever) being put into practice.

Comment: This type of counting problem is so common that a special symbol exists to describe the answer. The symbol 4!, read "four factorial", is shorthand for $4 \times 3 \times 2 \times 1$. In general, n! (n factorial) is the product of all of the integers from 1 to n. $1! = 1$, $2! = 2$, $3! = 5$, $4! = 24$, $5! = 120$, etc.

9. Permutations. There are 12 tones in one octave in the music most commonly in use in Western countries. Many twentieth century composers have used a modern style called twelve-tone music. In this music each of these tones is used once in a precise order to constitute a 12-note theme. Without considering rhythm, how many 12-note themes are possible?

Answer: Since there are only 11 notes possible after the first note is chosen, 10 notes after the first two, and so on, there are

$12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$, or 12!

in mathematical shorthand, or a little over 479 million themes possible.

Comment: The answer is technically called the number of permutations (arrangements) of 12 objects, 12 at a time, written ${}_{12}P_{12}$. The number of permutations of n things r at a time is given by the formula

$${}_nP_r = \frac{n!}{(n-r)!}.$$

For example, the number of 3-note themes with each note different is

$${}_{12}P_3 = \frac{12!}{(12-3)!} = \frac{12!}{9!} = 12 \times 11 \times 10 = 1320.$$

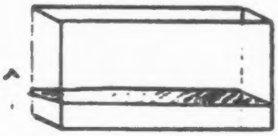
So many musical pieces begin with the same combination of 3 notes.

10. A liter is 1000 cm^3 (cubic centimeters). How many liters of water are needed to fill a fish tank which is 40 cm long, 25 cm deep, and 23 cm high?

Answer: The volume of a box is the product of its length, width, and height.

$$40 \text{ cm} \times 25 \text{ cm} \times 23 \text{ cm} = 2300 \text{ cm}^3,$$ which is 23 liters.

Comment:



It is accurate to think of the base of a box sweeping up (acting across) the height to achieve the volume. That is, volume = area of base x height. This idea is the basis of Cavalieri's principle in geometry and is applied in calculus to determine volumes of more complicated figures.

Comment: The exponent 3 in cm^3 signifies the dimension of volume, just as the 2 in cm^2 signifies area.

Multiplication Use Class C: Rate Factor

Suppose a box of diapers contains 18 diapers. How many diapers will a carton with 6 boxes of this size contain? The multiplication done to get the answer,

$$6 \text{ boxes} \times 18 \, \frac{\text{diapers}}{\text{box}} = 108 \text{ diapers}$$

exemplifies the use class we call rate factor. Here the rate is $18 \, \frac{\text{diapers}}{\text{box}}$. Other common rates are students per class, words per minute, pounds per square inch, milligrams per pill, and of course miles per hour. These rates appear in many real world multiplication situations. The form the multiplication takes may be analyzed by considering only the units and thinking of them as cancelling.

$$\frac{\text{unit 1}}{\text{unit 2}} \times \text{unit 2} = \text{unit 1}$$

$$\text{rate factor} \;\; \times \text{quantity} = \text{different quantity}$$

For example, a cubic foot of water weights about 62.4 lb. So 2.5 cubic feet would weigh

$$62.4 \, \frac{\text{lb}}{\text{ft}^3} \times 2.5 \text{ ft}^3$$

and the answer is in pounds, specifically 156.0 lb, more than most people would guess. (Waterbeds are very heavy.)

In some situations, two rate factors may be multiplied. For instance, if a team plays 30 home games, expects an average attendance of 6000 people per game, and sells tickets at $4 each, the expected revenue will be about

$$30 \text{ games} \times \frac{6000 \text{ persons}}{\text{game}} \times \frac{\$4}{\text{person}}$$

which when multiplied (with all units but dollars cancelling) is $720,000.

The rate factor use class is very rich in examples. Many of these examples, particularly when the rate factor is "over time", feel like acting across. Almost all word problems found in early work with multiplication can be categorized into this use class. So one could think of rate factor as being a use meaning of multiplication. However, this use class can also be thought of as being derived from rate division.

Examples:

1. Rate factors across time. A TV advertisement claims that, using a particular diet plan, a person can lose 2 pounds a week. How many pounds can a person lose in 5 weeks?

   Answer: total loss = (loss per week) x no. of weeks
   = 2 lbs/week x 5 weeks
   = 10 lbs.

   Comment: Almost everyone can do this kind of problem with small whole numbers "in his head". The point of the example is to illustrate the rate factor form, which also applies when the measures are not whole numbers. For instance, data could show that on the average a person loses 2.3 pounds a week, and one may want to know how many pounds would be lost in 5 1/2 weeks.

   Comment: Rates over time are seldom constant for long periods of time. One could not lose 2 pounds a week for long.

2. Speed. A trucker can drive 10 hours a day and the speed limit on the highway used is 55 mph (90 kph). Without speeding, what is the maximum distance a trucker can drive a single day? (In practice one would need to deduct time for fuel and rest stops.)

   Answer: distance = 55 mph x 10 hours = 550 miles

   or

   $$\text{distance} = 90\,\frac{\text{km}}{\text{hr}} \times 10 \text{ hours} = 900 \text{ km}$$

   Comment: Speed is by far the rate most familiar to students and can be used as a stepping stone to less familiar rates.

3. In a six-pack of cola, each can often holds 12 oz. How much cola is there altogether?

Answer: 6 cans x 12 $\frac{\text{oz}}{\text{can}}$ = 72 oz.

Comment: Books often ignore counting units, in this case "cans". This problem is then explained as 6 x 12 oz = 72 oz, and it looks like size change multiplication, with 6 as the scale factor. But it really is 6 cans, not a scalar, and we prefer classification as rate factor multiplication. For teaching, viewing the problem both ways is to be encouraged. In the size change sense, the amount of cola is sextupled. In the rate factor sense, the amount per can is distributed over 6 cans.

4. Cost. Pure gold was trading (in 1981) for between $395 and $550 per ounce. Suppose a ring contains the equivalent of .55 ounce of pure (24 karat) gold. At 1981 prices, what was the value of the gold in that ring?

Answer: At $395 per ounce, the value was 395 $\frac{\text{dollars}}{\text{ounce}}$ x .55 ounces = $217.25.

At $550 per ounce, the value was $550/ounce x .55 ounces = $302.50.

So the value was between $217.25 and $302.50.

Comment: Most students do not view the $ sign as representing a unit, because they are so accustomed to seeing unit designations *after* the numbers, as in 3 ft, 55 $\frac{\text{mi}}{\text{hr}}$, etc.

5. Rate factors over area. If carpeting sells for $22.95 a square yard including installation, how much will it cost to carpet a 9' x 12' room?

Answer: Total cost = (cost/area) x area

= $22.95/square yard x (3 yd x 4 yd)

= $\frac{\$22.95}{\text{square yard}}$ x 12 sq yds

= $275.40

Comment: \$22.95 a square yard equals \$2.55 a square foot, and the latter price appears less to many people, so is often used.

Comment: Carpeting usually comes in widths of 9', 12', or 15' so there would be little waste in this room. For other size rooms (for example, 11' x 11') one might need to buy more carpeting than is actually used.

6. Arrays. A movie theater has two aisles. There are 8 seats on the sides of the aisles and 12 seats in the middle in each row. If there are 25 rows in the theater, how many people can be seated?

Answer: $(8 + 12 + 8)\ \frac{\text{people}}{\text{row}} \times 25 \text{ rows} = 28\ \frac{\text{people}}{\text{row}} \times 25 \text{ rows} = 700$ people.

Comment: You could also find the answer by adding up the total number of seats in each section of the theatre. 8 x 25 on one side, 12 x 25 in the middle, and 8 x 25 on the other side. This gives a nice verification of the distributive property:

$$\begin{aligned} 8 \times 25 + 12 \times 25 + 8 \times 25 &= (8 + 12 + 8) \times 25 \\ &= 28 \times 25 \end{aligned}$$

7. Negative rates. A company is losing \$300,000 each month. At this rate how much would the company lose in a year?

Answer: $12 \text{ months} \times \frac{-\$300{,}000}{\text{month}} = -\$3{,}600{,}000.$

Comment: Negative numbers are usually avoided in problems of this type (see also Example 1 above). The next example illustrates that negatives cannot always be avoided so easily.

8. Negative and positive rates together. Last year the Colorado Ski Slope company lost an average of \$10000 each month from April to October, their off season, and made \$14000 per month during the colder months. This year they hope to cut their losses to \$8000 per warmer month and increase the profits to \$18000 per colder month. If they succeed, with how much profit will they finish the year?

Answer: $$\begin{aligned} &7 \text{ months} \times \frac{-\$8000}{\text{month}} + 5 \text{ months} \times \frac{\$18000}{\text{month}} \\ &= -\$56000 + \$90000 \\ &= \$34000 \end{aligned}$$

Comment: Were the profit to be negative, they would finish with a loss.

9. Conversion rates. How many mm are there in 3 meters?

Answer: Since 1000 mm = 1 m,

$$\frac{1000\text{ mm}}{1\text{ meter}} = 1$$

$$3\text{ meters} = 3\text{ meters} \times \frac{1000\text{ mm}}{1\text{ meter}} = 3000\text{ mm}$$

Comment: We call the fraction $\frac{1000\text{ mm}}{1\text{ meter}}$ a conversion rate.

Conversion rates share properties of measures and scalars.

In operations, their units work as if they are rates.

However, they all equal the scalar 1, so they can be exploited to change units in much the same way as we multiply by fractions of the form $\frac{n}{n}$ to form equivalent fractions.

10. Conversion within a system. How many feet are there in 5.3 miles?

Answer: Use the conversion rate $\frac{5280\text{ feet}}{1\text{ mile}} = 1$.

$$5.3\text{ miles} = 5.3\text{ miles} \times \frac{5280\text{ feet}}{1\text{ mile}} = 27984\text{ feet}$$

Comment: Compare Examples 8 and 9. One advantage of the metric system over the English system is that conversions within the metric system all involve powers of 10. Consequently they can often be done mentally.

11. Conversion between systems. What is the metric equivalent of 6'2"?

Answer: From the conversion 1 foot = .3048 meters, the conversion rate $.3048\ \frac{\text{meters}}{\text{foot}} = 1$.

$$6\text{ ft }2\text{ in} = 6\ 2/12\text{ ft}$$

$$\approx 6.166\text{ ft} \times .3048\ \frac{\text{meters}}{\text{ft}}$$

$$\approx 1.88\text{ meters}$$

Comment: In a book of tables, you might see

3.28 feet = 1 meter

Thus the number 1 equals either the ratio $\frac{3.28\text{ feet}}{1\text{ meter}}$ or its reciprocal $\frac{1\text{ m}}{3.28\text{ ft}} = \frac{.3048\text{ m}}{\text{ft}}$. Since the reciprocal of 1 is 1, the ratios are equal. The ratio to be used depends upon the direction in which you wish to convert.

Comment: Because conversion factors between systems are rarely whole numbers, people generally prefer to work in one system rather than convert between systems.

Comment: Some conversions, such as those between Fahrenheit, Celsius, or Kelvin temperatures, involve scales and cannot be done by multiplication alone.

12. If gasoline costs $1.25 a gallon and a car averages 21 miles per gallon, what is the cost of gasoline for a 300-mile trip?

Answer: $300\text{ miles} \times \frac{1\text{ gallon}}{21\text{ miles}} \times \frac{\$1.25}{1\text{ gallon}} = \$17.86.$

Comment: Notice the rate factor $\frac{1\text{ gallon}}{21\text{ miles}}$ is used in place of its more common equal $\frac{21\text{ miles}}{1\text{ gallon}}$, or 21 mpg.

Summary

Multiplication has three use classes, each of which is very rich in the breadth of its applicability. The size change use class encompasses problems involving "times as many", "part of", discounts, scaling, and a variety of other situations in which a quantity is multiplied by a scalar (the size change factor) to yield a second quantity. The acting across use class includes the calculation of area of a rectangle, the counting of ordered pairs or elements in arrays, and physical relationships in which one quantity acts across another to yield a product of a different sort than either given quantity. Rate factor uses cover situations in which one factor is a rate applied to the second factor.

Schoolbooks tend to explain away applications of multiplication as if they are all repeated addition and thus neglect the important and unique contributions of this operation.

Pedagogical Remarks

Size change. We have provided many examples for this use class because it is a rich real life use of multiplication that is generally neglected in schools.

Work with this use meaning can begin with the introduction to multiplication. When working with the "twos" facts, give examples of doubling and later give examples of halving; when working with the "threes" give examples of tripling and finding 1/3 of; etc. We have found that even primary school children understand "half of", and many understand "a third of" (Bell and Bell, 1982).

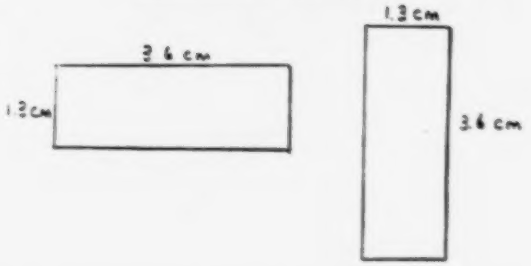
Some copy machines have settings allowing reductions to 5/6 or 2/3 of original size and magnifying glasses magnify 2.5 or 3 times. So the teacher can ask: If an object is 4 cm long to begin with, how big will it appear after being reduced or after being magnified?

Similar questions may be asked in discussions of percentages. Students should be able to answer such questions as "What is the effect of a factor of 50%?, "Is an error rate of 3% better or worse than an error rate of 2%?", "A 10% reduction in salary will leave a person with what percentage of the original salary?" These kinds of questions lead to understanding of the size change factor and a better understanding of multiplication. It may be better to introduce percentages as size change factors than as ratios. (We've seen no study one way or the other, but the poor understanding of percentages by students leads us to believe that introducing percentages as ratios is not particularly helpful.)

Acting across. Most children love games of chance; numbers or pairs or other combinations are often an issue in such games. For instance, if a person guesses at 2 multiple choice questions with 5 choices each, what is the chance that the person gets both correct? (Answer: There are 5 × 5 combinations of choices, so 1 chance in 25 of getting both correct.) School menus provide numbers of appetizers, main courses, and perhaps desserts. How many different meals are possible with a given menu?

Many students learn about areas of rectangles but never learn to associate these closely with multiplication (even though they multiply to get an answer). Area can be used to broaden one's understanding of multiplication as well as to provide a source of applications. The drawings below exhibit the use of area to confirm two properties involving multiplication.

Area interpretation

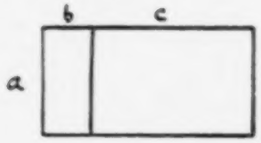


The two rectangles have the same area.

Multiplication meaning

$1.3 \times 3.6 = 3.6 \times 1.3$

Multiplication is commutative.



Area of whole = sum of areas of parts.

$a(b+c) = ab + ac$

Distributive property

Applications of areas abound: plots of land or water; material for making a box, clothes or curtains; carpeting or other floor coverings;

skins of animals; surfaces of plants or other objects. The area of a rectangle is basic to finding those areas even when they are not rectangular.

Some people are confused by the difference between area and length. Area and length are quite different: area measures the surface of a lake while length measures its shoreline; area measures the space inside a fence for a horse to roam, length measures the amount of fence needed.

This use class was the most difficult for us in terms of naming. The idea of this use class--one quantity acting through (or over, or across) a second quantity--may be more important than the specific examples.

Keeping units in the problem so that the student can see such things as

$$\text{kilowatts} \times \text{hours} = \text{kilowatt-hours}$$

is important in all instances of this use class. Pick examples that are part of a student's experience.

Rate factor. Of all the use classes of multiplication, this may be the most common. While discussion of such problems does not necessarily require division from the onset, the calculation of rates via division will have to occur quite early. Related multiplication and division facts can be reinforced by examples of rates and rate factors. A problem of the following type may be instructive:

> Fill in the blanks with numbers that work and write four mathematical relationships involving those numbers. Suppose a worker on an assembly line can complete ____ tables in an hour and works ____ hours a day. Then the worker can complete ____ tables in a day.

Typical answer: $7 \frac{\text{tables}}{\text{hr}}$, $8 \frac{\text{hours}}{\text{day}}$, $56 \frac{\text{tables}}{\text{day}}$. The four relationships are: $7 \times 8 = 56$; $8 \times 7 = 56$; $56 \div 8 = 7$; $56 \div 7 = 8$. Keeping the units in these relationships reinforces properties of fractions:

$$7\,\frac{\text{tables}}{\text{hour}} \times 8\,\frac{\text{hours}}{\text{day}} = 56\,\frac{\text{tables}}{\text{day}}$$

$$56\,\frac{\text{tables}}{\text{day}} \div 8\,\frac{\text{hours}}{\text{day}} = 7\,\frac{\text{tables}}{\text{hour}}$$

$$56\,\frac{\text{tables}}{\text{day}} \div 7\,\frac{\text{tables}}{\text{hour}} = 8\,\frac{\text{hours}}{\text{day}}$$

Repeated addition. Repeated addition is useful in early elementary school as a pedagogic link for a student who knows addition and needs to learn multiplication. However, repeated addition is possible only when one of the factors in the multiplication problem is a whole number, and it is convenient only when this factor is a small whole number, say less than 10. For instance, here are examples where repeated addition is a convenient procedure.

> (Size change) Jane works overtime for double her salary. If she makes \$3.25 per hour now, how much will she make after it is doubled?
>
> Answer: 2 × \$3.25/hr = \$3.25/hr + \$3.25/hr
>
> = \$6.50/hr
>
> (Rate factor) To calculate the distance when you have traveled 4 hours at 55 mph, add:
>
> 55 mi + 55 mi + 55 mi + 55 mi = 220 mi
>
> (Acting across) The area of a 4 by 5 rectangle can be found by noting that there are 4 columns of 5 squares each. 5 + 5 + 5 + 5 = 20

But the real world does not operate so simply. Repeated addition is made impossible if the actual numbers are slightly different: time-and-a-half for overtime for Jane; travel $4\frac{1}{4}$ hours; find the area of a

4.1 m by 5.6 m rectangle. Thus the use of multiplication in these situations cannot be explained by repeated addition, whatever the value of repeated addition is in explaining algorithms or remembering basic facts.

Since it is our fundamental rule that a use class should not be determined by the particular numbers involved in the use, we do not include repeated addition as a use class. When students are taught that multiplication is repeated addition, they tend to have great difficulty in dealing with the large number of multiplication situations that cannot be so interpreted.

Questions

1. A 1979 television commercial for Timex watches had the statement that these watches are 99.97% accurate. (a) In a day how much time might be lost or gained by a watch with that accuracy? (b) What percentage might indicate the inaccuracy of the watch?
2. An inch is now defined worldwide (including in the U.S.) as exactly 2.54 centimeters. Use this fact and multiplication by appropriate conversion factors to calculate exactly how many kilometers are in a mile.
3. In the proverbial Chinese restaurant, you are allowed one selection from column A and one from column B. If column A has 4 selections and column B has 5 selections, how many different combinations can be selected?
4. In a particular state there is a 4% sales tax. A store in this state is going out of business and advertises every item as 1/3 off. (a) What will it cost (including tax) for an item that originally sold for \$40.00? (b) To get the cost (including tax) of an item during this sale, by what single number could one multiply the original cost?
5. Give the dimensions of four different rectangles whose area is 15 m^2. ("Different" here means "with no dimensions alike".)
6. Jenny plans to work 8 weeks during the summer, $37\frac{1}{2}$ hours each week, and will earn \$5.25 per hour. What will be the total salary she will earn?

7. Call the local electric company, or use a recent electric bill, to find the cost of a kilowatt-hour of electricty. (This cost may vary depending on the time of year.) Use this cost to determine how much it costs to keep a 100-watt light bulb on for 24 hours.

8. What size change factor corresponds to each of the following readings on a gas gauge in a car? (a) 1/2 full; (b) full; (c) half full; (d) half empty; (e) empty.

9. Make up a multiplication situations in which one of the factors is in days and the product is in calories.

10. In order to attract customers, a store decides to sell a certain item A at a loss of 50¢ per item. (Such items are called "loss leaders".) Item B makes a profit of $4 per item. (a) If the store sells 300 of Item A and 50 of item B, how does it fare from these two items? (b) Show how this problem can be done with a negative rate factor.

Notes and Commentary

1. Misleading statements regarding multiplication
2. Other classifications of uses of multiplication
3. Sutherland's classification
4. Redundant use classes for multiplication
5. The connection with calculus
6. Dimensional analysis

1. Misleading statements regarding multiplication. Reputable methods books of previous generations described the operation of multiplication in ways which would make it hard, if not impossible, to assimilate any applications of the operation. Consider Morton (1937):

> "Multiplication is a short method for addition." (p. 224)

Morton is writing in an era in which there was preoccupation with questions of the order to teach whole number facts and the learning of whole number algorithms. His statement is not incorrect as a pedagogical device to help learn the small whole number multiplication combinations, but it is a disaster when one reaches fractions, decimals, or even large whole number factors.

Many students confuse uses of addition and multiplication. Could it be that they are influenced by some of the same confusions that are embodied in this quote from Wheat (1951):

> "Addition and multiplying both answer the same question 'How many altogether?' In both, we count quantities together into the same type of totals. The difference in the activities is in the sizes or amount of the quantities counted together and in the types of attention we give in advance of and while performing the activities." (p. 344)

Wheat is incorrect. If multiplication and addition answered the same questions, there would be no need to have two operations. They would be the same operation. After all, we do add very large numbers. One of the major messages that we hope arises from this volume is that the uses of multiplication are fundamentally different from those of addition and should be treated independently.

Generally, the search for simple pedagogies and the emphasis on counting seems to have led many authors to vastly oversimplify the uses of multiplication. Consider Grossnickel and Reckzeh (1973) with respect to this point.

> "Chapter 8 showed that the one problematic situation in addition consists in finding a sum when two addends are given. [Could there be any more circular description?--

> the authors.] The two problematic situations in subtraction consist in finding a missing addend when the sum and one addend are given and in finding how many more one number is than another number. [I.e., our recovering addend and comparison--the authors.] It is logical to conclude that there is one problematic situation in multiplication and two in division. A problematic situation in multiplication consists in finding the result of combining a given number of equivalent subsets of known size into one set. The following problem can be used to illustrate a multiplication situation: Find the cost of 5 stamps at 8¢." (pp. 190-91)

Grossnickle and Reckzeh then go on to analyze this problem as repeated addition. Thus they miss virtually every problem situation of multiplication!

2. *Other classifications of uses of multiplication*. Not all authors have been guilty of the oversimplifications referred to in note 1. Kennedy (1970) gives four "approaches to teaching mutiplication" of whole numbers: repeated addition, array, Cartesian product, and ratio. Swenson (1973) lists "six ways of inerpreting multiplication": counting by equal-sized intervals, high-powered addition of equal addends, ratio-to-1, array, Cartesian product, and union of equivalent disjoint sets.

These authors, and also Kansky (1969) and Vest (1968) in their more extensive analyses, have employed for pedagogic purposes more than one type of meaning of an operation, sometimes considering mathematical properties, sometimes representations, sometimes uses.

Usiskin (1976) uses Cartesian product, area, size change, and repeated addition.

3. *Sutherland's classification*. Sutherland's analysis of word problems in grades 3-6 found eight general patterns of multiplication and she placed these into five categories, as follows:

> Patterns requiring the total amount to be found:
> 1. Given the number or amount in one group and the number of groups, find the total number or amount.
> 2. Given the cost or amount of one unit and the number of units, find the total cost or amount of money.
>
> Patterns requiring the total distance and involving the concepts of rate and time:
> 3. Given the rate per unit of time and the time, find the total distance.
> 4. Given the mileage per gallon of gasoline and the number of gallons of gasoline, find the total distance.

Patterns requiring that a fractional part of a number be found:

5. Given the number, find a fractional part of it, the numerator of the fraction being 1. (considered as multiplication in grades 5 and above, division before that)
6. Given a number find a fractional part of it, the numerator of the fraction being greater than 1 or the fraction expressed as a decimal.

Pattern requiring a per cent of a number:

7. Given the number, find a per cent of it.

Pattern requiring the whole when a part is given:

8. Given one part of the number, find the total number.

Her analysis found that the word problems in grades 3-6 were ovewhelmingly from patterns 1 and 2; there being the following numbers in each pattern at these levels.

Pattern	1	2	3	4	5	6	7	8
Appearances	1202	2343	98	9	294	167	202	44

Her patterns 1-4 correspond to our rate factor; we consider them to differ only in context. Her patterns 5-7 fit our size change, the difference between 5 and 6 being only the size of number and between those and pattern 7 being only the way the number is represented. Neither of these differences to us determines a different use. Her pattern 8 does not correspond to any of our multiplication patterns. Here is a problem she gives.

> Nancy paid 10¢ for 1/4 lb. of butter. At this rate, how much would 1 lb. of butter cost?

Because the answer is not given by multiplying the numbers, but by dividing 10¢ by 1/4 lb, we classify the problem as division, specifically as rate division (cost per pound).

Instances of acting across are not found by Sutherland in the books she examined. We do not know if the same would be true today. If so, students are missing a large category of the uses of multiplication.

4. Redundant use classes for multiplication. We troubled long over the question of inclusion of repeated addition as a use class. The arguments for inclusion are that (1) it is familiar to everyone and (2) some uses of whole numbers seem to naturally fall in it. The arguments against are that (1) repeated addition is merely a computational shortcut and introduces no new situations not covered by addition, (2) repeated addition only works when at least one factor is a whole number, and (3) anything one would call repeated addition fits one of the other use classes. Further discussion is found under the pedagogical comments for this chapter.

We considered a use class entitled conversion factor, including questions like Examples 9-11 of Section C. The argument in favor of a separate use class involves the dual nature of conversion factors; they are unitized quantities with a value equal to the scalar 1. Thus conversion factors share properties of rate factors and size change factors. We decided that conversion factors act and feel like rate factors. (E.g., we say that there are 3 feet per yard or 3 feet for each yard.) So, despite their importance and their treatment in books as a separate topic, we categorized conversion factors as a special type of rate factor.

In early versions of these materials, we distinguished the physical examples of acting across from area and both of these from Cartesian product, with three separate use meanings:

(1)	Cartesian product:	If set A has a elements and set B has b elements, then there are a × b ordered pairs of elements whose first element is from A and second element is from B.

We called this pairing or ordered pair. Examples 5-9 of Section B fit this designation.

(2)	Area:	The area of a rectangle with length a and width b is a × b.

Examples 1, 2, and 10 of Section B fit area. Volume is an extension.

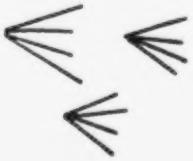
(3)	Acting across:	First quantity × interval or second quantity through which first quantity is acting = product outcome.

The product was work, energy, pressure, power, or other physical combinations. Examples 3 and 4 of Section B fit this.

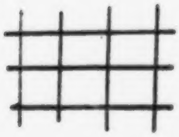
It was obvious to us that area is a special instance of acting across. In calculus this allows area to be used as a manifestation of many physical relationships (see note 5 below). But area is so fundamental geometrically and pedagogically that others may wish to elevate it into a use meaning of its own.

Less obvious was the subsuming of Cartesian product under acting across. Our reasoning was as follows. Cartesian products are usually represented by tree diagrams, intersections of line, lattice point arrays, and are defined as ordered pairs of elements of sets. In all of these conceptualizations, each element in a first set is linked to every element in a second set and in that way the first set "acts across" the second set. The lattice point representation suggests the rectangular arrays of squares one confronts in first discussions of area.

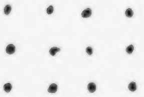
BEST COPY AVAILABLE



tree



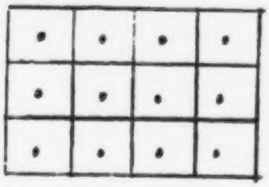
intersection



array

$$\{(1,a),\ (1,b),\ (1,c),\ (1,d)\ (2,a),\ (2,b),\ (2,c),\ (2,d)\ (3,a),\ (3,b),\ (3,c),\ (3,d)\}$$

definition



array → area

Perhaps due to the way we teach or have been taught, the size change instances with integer size change factors (specifically, "times as many") feel different than those with fractional values ("part of"). For instance, "5 times as many" feels different than "2/3 of". We thought of partitioning the size change examples by the values of the size change factors, but this would have violated a fundamental organizing principle of this volume, that use meanings are independent of value of numbers.

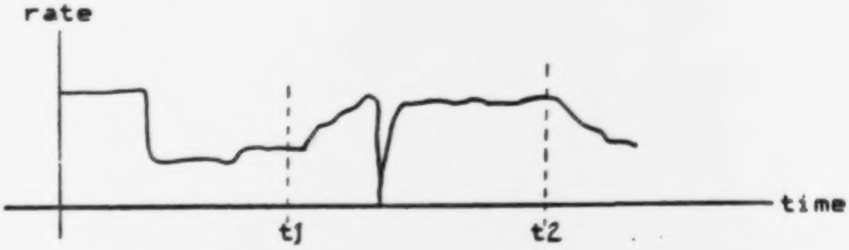
5. The connection with calculus. The area use of multiplication (a special type of acting across) explains why integration in calculus has so many applications. The formula

$$A = lw$$

differs only in letters from such physical formulas as

$d = rt$ (distance = rate x time)
$F = mA$ (Force = mass x acceleration)
$M = fd$ (Example 3, Setion B)
$P = VI$ (Example 4, Section B)
or $E = IR$ (Ohm's law).

By letting the factors on the right hand side of these equations be represented as the length and width of rectangles, distance, force, moments, power, or electrical pressure can be represented as an area. For instance, if the curve below pictures a rate function, then, because height x width = area, the distance travelled from time t1,



to time t2 is the area under the curve between the lines x = t1 and x = t2. One thinks of the rate acting across time to produce distance. (See also Usiskin, 1976.)

6., *Dimensional analysis*. We are influenced in this essay by Freudenthal, who in his book *Mathematics as an Educational Task* writes,

> "The argument of rigour against computations with concrete numbers (what we have called quantities) is completely mistaken. Concrete numbers are absolutely rigorous, and the resistance of some mathematicians to them is sheer dogmatism." (p. 207)

Freudenthal then goes on to give a brief sketch of the formal theory.

Our theory is not presented in as symbolic a way as Freudenthal's because we wish less mathematically sophisticated readers to understand the properties of quantities. The properties, with examples, are given below.

(1) Two quantities with the same units (counting, measure, or monetary) may be added or subtracted. (However, the addition may not always be appropriate or meaningful.)

Examples: 3 oranges + 4 oranges = 7 oranges
33.2 cm + 146 cm = 179.2 cm
$100 - $47 = $53

Symbolically, using a and b to stand for numbers and U to stand for the unit, we may write:

$$aU + bU = (a + b)U$$
$$aU - bU = (a - b)U$$

From this, one can prove that $0U = 0$. That is, zero of any unit can be treated as the number zero. Note that zero may have other meanings with labels other than units (for example, 0°C does not equal the number 0).

(2) If one unit is a multiple of another unit, then the quotient of the corresponding quantities is the number 1.

Examples: 12 inches = 1 foot, so we may write

$$\frac{12\text{ inches}}{1\text{ foot}} = 1 \quad\text{or}\quad \frac{1\text{ foot}}{12\text{ inches}} = 1.$$

1 kilometer = 1000 meters, so

$$\frac{1\text{ kilometer}}{1000\text{ meters}} = 1 \text{ and } \frac{1000\text{ meters}}{1\text{ kilometer}} = 1.$$

1 kg = 2.2 pounds approximately, so

$$\frac{1\text{ kg}}{2.2\text{ lb.}} = 1 \text{ approximately and } \frac{2.2\text{ lb.}}{1\text{ kg}} = 1$$

approximately.

Symbolically, if

$$1\ M = k\ L, \text{ then } \frac{1}{k}\,\frac{M}{L} = 1 \text{ and } \frac{k}{1}\,\frac{L}{M} = 1$$

(3) Quantities may be multiplied. A new derived unit is formed.

Examples: Suppose we have a rectangular carton with dimensions 80 cm, 60 cm, and 100 cm. The area of one of its faces is 4800 cm^2 (4800 square centimeters). The volume of the carton would be 480,000 cm^2 (480,000 cubic centimeters). The notation with exponents reflects the multiplication of the units.

Electricity is commonly measured in kilowatt-hours (kwh). This unit is the product of multiplying kilowatts by hours.

Symbolically, $a\ L \times b\ M = ab\ LM$.

(4) A quantity may be multiplied by a scalar.

Examples: An item costing $50 is subject to a 20% discount. 20% is a scalar; the resulting discount is 20% x $50 or $10. What is generally unnoticed is that the dollar sign is carried through the computation.

A piece of furniture 8 feet long will have length 1/16 of that when represented in a doll house. The length wil be 1/16 x 8 feet = 0.5 feet.

In symbols, a(b L) = (ab) L.

(5) Quantities with different units may be divided, forming a new derived unit, called a rate.

Examples: If you travel 50 miles in 5 hours, the expression

$\frac{50 \text{ miles}}{5 \text{ hours}}$ means 10 miles per hour.

Dividing the other way yields an equal quantity.

The expression $\frac{5 \text{ hours}}{50 \text{ miles}}$ means 1/10 hours per mile.

The new derived units are miles per hour and hours per mile.

In symbols, $\frac{a}{b}\frac{L}{M} = \frac{a}{b}\frac{L}{M} = \frac{a}{b}$ L per M.

(6) The division of two quantities with the same units yields a scalar.

Example: Four hundred miles is twice as far as 200 miles. That is,

$\frac{400 \text{ miles}}{200 \text{ miles}} = 2$

In symbols, $\frac{a}{b}\frac{L}{L} = \frac{a}{b}$.

(7) Multiplication and division of quantities may be combined.

Examples: Suppose you travel 2 1/2 hours at 5 kilometers per hour. Since distance = rate x time, distance = 5 km/hr x 2 1/2 hr, or distance = 12 1/2 km.

In college it is estimated that a student reads 5 books for each course. How many books will be read by a class of 350 students if each student takes 4 courses?

$$5 \frac{\text{books}}{\text{course}} \times 4 \frac{\text{courses}}{\text{student}} \times 350 \frac{\text{students}}{\text{class}}$$

$$= 7000 \frac{\text{books}}{\text{class}}$$

Symbolically, $a\,L \times \frac{b}{c}\,\frac{M}{L} = \frac{ab}{c}\,M$.

(8) Dividing by a rate is equivalent to multiplying by its reciprocal.

Examples: 1800 students in classrooms averaging 25 students/classroom requires

$$\frac{1800 \text{ students}}{25 \,\frac{\text{students}}{\text{classroom}}} = 1800 \text{ students} \times \frac{1}{25}\,\frac{\text{classrooms}}{\text{student}}$$

$$= 72 \text{ classrooms.}$$

A lawyer makes $75,000/year and works 250 days/year. What is the average earnings per day?

$$\frac{\frac{\$75000}{\text{year}}}{250\,\frac{\text{days}}{\text{year}}} = \frac{75000}{250}\,\frac{\$}{\text{year}} \times \frac{\text{year}}{\text{days}}$$

$$= 300\,\frac{\$}{\text{day}}$$

$$= \$300 \text{ per day.}$$

In symbols, $\frac{a}{b}\,\frac{L}{\frac{M}{N}} = \frac{a}{b}\,L \times \frac{N}{M} = \frac{a}{b}\,\frac{LN}{M}$.

If L = M, they cancel, as above.

Because some units can be added, subtracted, multiplied, or divided, it looks like anything can be done with units. The final properties of unit arithmetic show the limits of the arithmetic applications of operations with unit quantities.

(9) Two quantities with different labels cannot be added or subtracted.

Examples: You cannot add 3 apples to 4 oranges unless the labels are modified to be the same (fruit).

One would not add "miles going" to "miles coming back" unless one was interested in a label broad enough (miles traveled) to include both.

(10) A scalar cannot be added to or subtracted from a quantity.

Example: 3 kg + 5 has no meaning.

The arithmetic of unit quantities makes it possible to analyze applications by looking only at the units. This type of analysis is known in science classes as dimensional analysis. There is no doubt that dimensional analysis can clarify certain situations. One nice example is found in the Random House Encyclopedia, page 1450:

> If the volume of a paraboloid were stated to be H^2/(8D), with H as its height and D its base diameter, then without any calculations at all a student can be sure that the formula is wrong. It involves the product of two lengths divided by a length and so has dimensions L^2/L or L. It must therefore represent a length; it cannot possibly represent a volume. (The correct formula is (H^2)D/8.)"

CHAPTER 8

USES OF DIVISION

Division has two basic use meanings: rate and ratio. The use classes for division are derived from these meanings and the relationships division has with multiplication. We recognize five use classes:

A. Ratio
B. Rate
C. Rate divisor
D. Size change divisor
E. Recovering factor

Division Use Class A: Ratio

Suppose it takes two partners 45 minutes and 10 minutes to commute to work from their homes. We can compare these times by either subtraction or division.

By subtraction: 45 min - 10 min = 35 min.

By division: $\frac{45 \text{ min}}{10 \text{ min}} = 4.5.$

We can say that it takes one partner 35 minutes longer or 4.5 times as long to get to work than the other partner.

This second type of comparison is called ratio comparison or simply ratio.

> Ratio comparison use meaning of division
>
> Let a and b be quantities with the same labels. Then, a divided by b, written $\frac{a}{b}$, is a ratio which compares a to b.

Ratio comparison requires that the quantities have the same labels. If one partner's commute was measured as $\frac{3}{4}$ hr, we would change that to 45 min before dividing. Ratios can be formed by dividing either number by the other. In the above situation, dividing in the other order,

$$\frac{10 \text{ min}}{45 \text{ min}} = \frac{10}{45} \approx \frac{1}{4},$$

and we could say that it takes the first partner about $\frac{1}{4}$ as long to get to work as the second.

All percentages, all probabilities, some fractions, and many of the scalars found in the discussions of previous chapters (e.g., size change factors in size change multiplication) can be interpreted as ratios.

This section marks the third idea to which the word ratio applies. First is as a use of numbers, (Chapter 1, Section D), ratio comparison, in which a ratio stands for a single number. Second is as an ordered pair, a:b, as discussed in Chapter 2, Section A. Third is as a use meaning of an operation, as exemplified in this section. These three uses of "ratio" have caused linguistic and conceptual confusion and difficulty for students and educators. An extended discussion of the issues is given in note 8.

Examples:

1. Percentage of discount. If an item costing \$30 is reduced \$6, what is the percentage of discount?

   Answer: $\frac{\$6}{\$30} = .20 = 20\%$, so the discount is 20%.

   Comment: Some books set up a proportion $\frac{6}{30} = \frac{x}{100}$ to help answer this problem. This strikes us as unnecessarily sophisticated machinery for a problem which requires only a single operation and rewriting for its solution.

2. Growth rate. The world population is estimated to have grown from 1.1 billion in 1850 to 1.6 billion in 1900, and to 2.5 billion in 1950. In which of these 50-year periods was the growth rate higher?

   Answer: The growth rate can be found by comparing the populations by division. For the years 1850 to 1900, divide as follows:

   $$\frac{\text{1900 population}}{\text{1850 population}} = \frac{\text{1.6 billion}}{\text{1.1 billion}} \approx 1.45$$

   which signifies that the population in 1900 was 145% of the population in 1850. That is called a growth rate of 45%.

   For the years 1900-1950, do the corresponding division.

   $$\frac{\text{1950 population}}{\text{1900 population}} = \frac{\text{2.5 billion}}{\text{1.6 billion}} \approx 1.56$$

This signifies a 56% growth rate in the years 1900-1950, so the population was growing at a faster rate in these years.

Comment: The growth "rates" in this example are over a 50-year period and could not be compared to growth rates in periods of other length without calculations to adjust for the difference in time. Specifically, it would be incorrect to divide these percentages by 5 to get a growth rate for a decade.

Comment: If populations at two different times were equal, the ratio would be 1 or 1.00, signifying a 0% growth rate, which agrees with the common term "zero population growth".

Comment: The amount of growth is found by subtraction comparison. The amount of growth from 1900 to 1950 was almost double that from 1850 to 1900 (.9 billion vs. .5 billion), but the rate of growth was only about 11% higher.

3. Interest rate. A bank adds $1.99 to an account after three months as interest on a bank balance of $159.35. What quarterly interest rate is this?

Answer: $\frac{\$1.99}{\$159.35} \approx .012488$, or approximately .0125. This would seem to indicate that the rate is 1.25% per quarter (three month period).

Comment: Banks call this rate 5% yearly, although, if the interest were left in the account, the earnings at the end of a year would be slightly greater than 5% due to compounding.

4. Probability. A die is tossed. What is the probability that the die will show a number that is a five?

Answer: It is natural to assume that each of the six faces of the die is equally likely to turn up. Then the probability is:

$$\frac{\text{number of faces with a 5}}{\text{total number of faces}} = \frac{1}{6}$$

Comment: Though not every probability is calculated by dividing, every probability can be interpreted as the result of a ratio comparison. For instance, though a precipitation probability of 70% means that 7 of 10 times you should expect precipitation, these probabilities are calculated using sophisticated mathematical models.

5. Scale. Cadillacs in the early 60's were 20 feet long. What is the scale of a model of one of these cars if the model is 6" long?

Answer: $\frac{6''}{20'} = \frac{6''}{240''}$, so the model is $\frac{6}{240}$ actual size. In using a fraction to give the comparison, most people try to make the numerator as small a whole number as practical. Here $\frac{6}{240}$ would be simplified to $\frac{1}{40}$ and the model called $\frac{1}{40}$ actual size.

Comment: The answer might also be written as the ratio 1:40 to signify the scale of the model.

6. How many times as many? About how many times as many people live in the United States (population 220 million in 1978) as in Morocco (population 18 million in 1978)?

Answer: $\frac{220}{18} \approx 12.2$, so about 12 times as many.

Comment: Since we knew that the "millions" would cancel, it was not necessary to use them in the calculation.

7. Percentage. A 12-mile stretch of highway is to be repaved. If 9 miles have been finished, what percentage of the job has been done?

Answer: $\frac{9 \text{ miles}}{12 \text{ miles}} = .75 = 75\%$

Comment: We could also say that $\frac{3}{4}$ of the paving has been done. Any of $\frac{9 \text{ miles}}{12 \text{ miles}}$, $\frac{9}{12}$, $\frac{3}{4}$, .75, or 75% may be the most appropriate way of answering the question, depending on the reason for asking the question.

8. Change. Profits of a large company are 30 million dollars one year and 75 million dollars the next. How can the change be described?

Answer: $\frac{\$75 \text{ million}}{\$30 \text{ million}} = \frac{5}{2} = 2.5 = 250\%$.

One could say: (a) Profits were $\frac{5}{2}$ those of a year before.

(b) Profits were 2.5 times those of a year eariler.

(c) Profits were 250% of those a year earlier. (If they had made the same amount, it would be 100%.) (d) Profits increased 150%.

Comment: Contrast profits being "250% of last year's" and "a 250% increase from last year's". An increase of 250% would mean this year's profit was 350% of last year's, or 105 million dollars.

Comment: Of course one could subtract 30 from 75 and say there was a 45 million dollar increase. The company could use either type of comparison in its reports and still be correct.

9. Compare the U.S. unemployment rates of 5.8% (April, 1979) and 10.5% (October, 1982).

Answer: Comparing by division, $\frac{10.5\%}{5.8\%} = 1.81$, so the unemployment rate was 81% higher in 1982 that it was $3\frac{1}{2}$ years earlier.

Comment: Subtracting, 10.5% - 5.8% = 4.7%, and one could also say that the unemployment rate was 4.7% higher in 1982 that it was 3 1/2 years earlier. The English language does not distinguish between the different ways that 81% and 4.7% were calculated. Thus, when you hear of a percentage increase or decrease, it could have been calculated either by subtraction or division. This ambiguity can be used to deceive the unwary.

Comment: Note here that two ratios are being divided, yielding a third number used as a ratio.

Division Use Class B: Rate

Division allows comparison of counts or measures with dissimilar labels. Then a rate is formed. The first example of rate that may come to mind is speed. Going 125 miles in $2\frac{1}{2}$ hours is an average speed of 50 mph.

$$\frac{125 \text{ miles}}{2.5 \text{ hours}} = \frac{125}{2.5}\ \frac{\text{miles}}{\text{hr}}$$
$$= 50\ \frac{\text{mi}}{\text{hr}} \text{ or } 50 \text{ mph}$$

Another common rate is unit cost. For example, if 6 bottles of soft drink cost \$1.59, then

$$\frac{\$1.59}{6 \text{ bottles}} = \$.265/\text{bottle} = \$.27/\text{bottle}$$

Keeping track of units is essential with rates. Here the unit is "dollars per bottle". Similarly, if 159 students are in 6 classes, there are 27.5 students/class, and the unit is "students per class".

Examples:

1. Speed. What is the average typing speed if 400 words are typed in 6 minutes?

Answer: $\frac{400 \text{ words}}{6 \text{ minutes}} = 67\ \frac{\text{words}}{\text{minute}}$

Comment: The answer is read "67 words per minute". The word "per" usually signifies a division rate situation. Though it has other contexts (per se, per instructions, etc.), it is perhaps the English word most exclusively associated with a specific operation. (Other words, like "of" or "times", are more often used in many non-mathematical ways.)

Comment: Dividing in the other order also is meaningful.

$$\frac{6 \text{ minutes}}{400 \text{ words}} = \frac{1}{67}\ \frac{\text{minutes}}{\text{word}},$$

or about 1 second per word. This is another way of describing the typing speed.

2. Birth rate. In 1979, there were 3,383,000 live births in the U.S. population of 219,000,000. What birth rate is this?

Answer: $\frac{3,383,000 \text{ births}}{219,000,000 \text{ persons}} = .0154\ldots$ births/person,

which is usually converted by multiplying numerator and denominator by 1000 to 15.4 births per 1000 population.

Comment: Neither 0.154 births or 15.4 births is possible, yet 0.154 births/person and 15.4 births per 1000 population have definite meaning. This signifies that the numbers 0.154 and 15.4 in the answer must be attached to the rate units and not to counting units.

3. Average class size. The 6 seventh grade classes in a school have a total of 131 students. What is the average number of students per class, the average class size?

Answer: $\frac{131 \text{ students}}{6 \text{ classes}} = 21.8$ students per class.

Comment: The notion of 21.8 students is impossible, but the unit of the answer is not "students" but "students per class".

4. Density. New York City (1970) had a population of 7.8 million people living in an area of 300 square miles. Boston's 700,000 people lived in 46 square miles. Which city was more densely populated?

Answer: For New York: $\frac{7,800,000 \text{ people}}{300 \text{ sq mi}} = 26,000$ people per sq mi.

For Boston: $\frac{700,000 \text{ people}}{46 \text{ sq mi}} = 15,000$ people per sq mi.

New York was more densely populated.

Comment: From almanac information we conclude that in 1970 New York City was by far the most densely populated large city in the United States. The most densely populated part of New York City was Manhattan, with almost 68,000 residents per square mile. Many U.S. cities have a density between 2000 and 4000 people per square mile.

5. Rates of change. A child grows 5 inches in 3 years. What is the rate of change of the height in this time interval?

Answer: $\frac{\text{change of height}}{\text{time}} = \frac{5 \text{ inches}}{3 \text{ years}} = 1\frac{2}{3}\ \frac{\text{in}}{\text{yr}}$.

Comment: The division to calculate a rate can be made in either order. The numerical values will be reciprocals but the units will be reversed to compensate. Dividing the other way, $\frac{3 \text{ years}}{5 \text{ inches}} = \frac{3}{5}$ yr/in., indicating that, on the average, it took $\frac{3}{5}$ of a year for the child to grow an inch.

6. Growth rate. 1001 Questions Answered About Trees tells us that one can almost see a thriving bamboo tree grow, for such a tree can grow as much as 18 inches in a single day. What is the average growth per hour of a bamboo tree growing at that rate?

Answer: $\frac{18 \text{ inches}}{1 \text{ day}} = \frac{18 \text{ inches}}{24 \text{ hours}} = \frac{3}{4}$ inch per hour.

Comment: This is very fast, one of the fastest rates for any tree, but then that's why it's an interesting piece of information.

7. Unit cost. If 20 cookies cost 40¢, what is the cost of one cookie?

Answer: $\frac{40¢}{20 \text{ cookies}} = 2 \frac{¢}{\text{cookie}}$.

Comment: If the situation had 80 cookies costing 40¢, most people would probably divide the other way,

$$\frac{80 \text{ cookies}}{40¢} = 2 \frac{\text{cookies}}{\text{penny}} .$$

This is a manifestation of the preference for whole numbers rather than fractions.

8. Partition. The job of polling the 2500 residents of a town is to be split up among 40 interviewers. On the average, how many people will each interviewer poll?

Answer: Think of looking for "people per interviewer". This indicates the appropriate order of division.

$$\frac{2500 \text{ people}}{40 \text{ interviewers}} \approx 63 \text{ people per interviewer.}$$

Comment: The quotient is exactly 62.5, indicating that each interviewer could interview 62 people and half the interviewers could interview one more (for a total of 63) to reach all residents.

9. Speed. In a half hour, a jogger was able to complete 11 laps around a quarter-mile track. What was the jogger's average speed (in miles per hour)?

Answer: The unit in the rate, miles per hour, indicates to divide the number of miles by the number of hours, i.e., distance by time. The distance was 11 laps x $\frac{1/4 \text{ mile}}{\text{lap}} = \frac{11}{4}$ miles.

$$\begin{aligned}\frac{\frac{11}{4}\text{ miles}}{\frac{1}{2}\text{ hour}} &= \frac{11}{4} \div \frac{1}{2}\text{ mph} = \frac{11}{2}\text{ mph},\\ &= 5\frac{1}{2}\text{ mph, the average speed.}\end{aligned}$$

Comment: When quantities are measured in fractions, rate questions can lead to division of fractions.

Comment: Dividing the other way:

$$\begin{aligned}\frac{\frac{1}{2}\text{ hour}}{\frac{11}{4}\text{ miles}} = \frac{30\text{ minutes}}{\frac{11}{4}\text{ miles}} &= 30 \times \frac{4}{11}\,\frac{\text{minutes}}{\text{mile}}\\ &= \frac{120}{11}\,\frac{\text{minutes}}{\text{mile}}\\ &= 10\frac{10}{11}\,\frac{\text{minutes}}{\text{mile}}\end{aligned}$$

10. Conversion rates. On May 23, 1980, 1 German mark was worth approximately \$.56. How does this information enable one to convert back and forth from dollars to marks?

Answer: Since 1 mark = \$.56, $\frac{\$.56}{1 \text{ mark}} = 1$ and $\frac{1 \text{ mark}}{\$.56} = 1$, so the conversion rate is either .56 $\frac{\text{dollars}}{\text{mark}}$ or $\frac{1}{.56}\,\frac{\text{marks}}{\text{dollar}}$.

Comment: Thus to convert from marks to dollars, multiply by .56. To convert from dollars to marks, multiply by $\frac{1}{.56} = 1.7857...$, or divide by .56. Current rates can be found in the financial pages of many newspapers.

11. Fuel performance. If a car travels 300 miles after a fill-up, and 14.3 gallons were needed to fill up the tank again, what gasoline mileage (i.e., miles per gallon) is the car getting?

    Answer: $\frac{300 \text{ miles}}{14.3 \text{ gal}} \approx 21$ miles per gallon (mpg)

    Comment: In the United States, gas consumption is measured in miles per gallon. In other parts of the world, gas consumption is usually measured in other units, e.g., in milliliters per kilometer or kilometers per 1000 liters. In the unit $\frac{\text{ml}}{\text{km}}$, lower values indicate more efficient gas usage, the opposite of the case with mpg or $\frac{\text{km}}{1000 \text{ liters}}$.

12. Acceleration. A car is going 10 mph to begin with and its speed increases to 12.4 mph after one second and 16 mph after another second. Calculate its acceleration in each second.

    Answer: In the first second, its acceleration is 2.4 mph per second. Since "per hour per second" mixes units, one changes miles per hour to feet per second (see comment) and gets 3.52 feet per sec^2. In the second second, its acceleration is 3.6 mph per second, or 5.28 feet per sec^2.

    Comment: To convert mph to feet per sec, do the following.

    $2.4 \text{ mph} = 2.4 \frac{\text{miles}}{\text{hr}} \times \frac{5280 \text{ ft}}{\text{mile}} \times \frac{1 \text{ hr}}{3600 \text{ sec}} = 3.52 \frac{\text{ft}}{\text{sec}}$

    Thus $\frac{2.4 \text{ mph}}{\text{sec}} = \frac{3.52 \text{ sec}}{\text{sec}} = 3.52 \frac{\text{ft}}{\text{sec}^2}$, often read as "3.52 feet per second per second".

13. Changes in rates. From August 1-15, 1967 to August 1-15, 1970, average rates on business loans in the U.S. increased from 5.95% to 8.50% per year. From February to May 1974, these rates increased 9.91% to 11.15% per year. In 1976, they were back below 8%, so you can see that these rates are quite volatile. In which of the earlier periods was there greater acceleration in the cost of borrowing money?

Answer: $\text{acceleration} = \dfrac{\text{change of rate}}{\text{time}}$

8/67 to 8/70:

$$\text{acceleration} = \frac{\frac{8.50\%}{\text{year}} - \frac{5.95\%}{\text{year}}}{3 \text{ years}} = \frac{2.55\%}{3\ \text{yr}^2} = 85\% \text{ per year per year}$$

2/74 to 5/74:

$$\text{acceleration} = \frac{\frac{11.15\%}{\text{year}} - \frac{9.91\%}{\text{year}}}{3 \text{ months}} = \frac{1.24\%}{\frac{1}{4}\ \text{yr}^2} = 4.96\% \text{ per year per year}$$

Comment: To avoid the phrase "per year per year", synonyms are used. The second acceleration might be described as "the equivalent of a yearly increase of 4.96% in the annual rate".

Division Use Class C: Rate Divisor

Here is a problem many people can do "in their head". If one can average 50 mph on a trip, how long will it take to travel 550 miles? The answer, 11 hours, can be found by dividing. When this division is analyzed, the divisor is a rate, and the computation--even with units--follows the rules for division of fractions.

$$\frac{550\text{ miles}}{50\,\frac{\text{miles}}{\text{hr}}} = \frac{550}{50}\;\frac{\text{mi}}{\frac{\text{mi}}{\text{hr}}} = 11\text{ hr} \times \frac{\text{hr}}{\text{mi}} = 11\text{ hr.}$$

Many textbooks introduce division through questions involving a set of objects which is to be split into groups of the same size. Here is an example:

> There are 40 objects to be split into groups so that there are 5 objects in each group. How many groups will there be?

Analyzing this kind of problem, we again see that the divisor is a rate.

$$\frac{40\text{ objects}}{5\,\frac{\text{objects}}{\text{group}}} = 8\text{ groups.}$$

Rate divisors are quite common, but seldom acknowledged, perhaps due to the complexity of the units.

Examples:

1. Having borrowed \$1500, a person wishes to pay back \$50 a month. How long will this take?

   Answer: $\frac{\$1500}{\$50/\text{month}} = 30$ months

   Comment: Most lenders would require interest on a loan, so 30 months at \$50 would not pay off a \$1500 loan.

   Comment: Most people would ask how many \$50 payments would make \$1500, assuming months all along. That avoids the rate unit in the divisor.

2. At a rate of 70 words per minute, how many minutes of typing should it take a secretary to type a 50,000 word manuscript?

Answer: $\dfrac{50{,}000 \text{ words}}{70\,\dfrac{\text{words}}{\text{minute}}} = 714 \text{ minutes}$

Comment: Due to necessary breaks, page changes, error corrections, etc., one cannot reliably count on more than about 40 minutes of actual typing per hour. At 40 minutes of typing per hour, we can calculate how many hours it will take to type the manuscript, again by a rate divisor division.

$$\dfrac{714 \text{ minutes}}{40\,\dfrac{\text{minutes}}{\text{hour}}} = 18 \text{ hours}$$

3. How many LP's at $7.99 can be bought for $50?

Answer: $\dfrac{\$50}{\dfrac{\$7.99}{\text{LP}}} = 6.25\ldots \text{ LP's} = 6 \text{ LP's with some change.}$

Comment: For computational convenience $7.99 may be rounded to $8. Only if the answer came close to an integer would one have to question whether this rounding changed the quotient.

Comment: With tax in some places only 5 LP's could be purchased.

4. If the average gas mileage in highway driving for a car is 30 mi/gal, about how much fuel will it use on a 200 mile trip?

Answer: $\dfrac{200 \text{ mi}}{30 \text{ mi/gal}} = 6.\overline{6} \text{ gal, or about 7 gallons.}$

Comment: One could also conceptualize this problem as the quotient of two rates. Again the division of units is like division of fractions.

$$\dfrac{200\,\dfrac{\text{mi}}{\text{trip}}}{30\,\dfrac{\text{mi}}{\text{gal}}} = 6.\overline{6}\,\dfrac{\text{mi}}{\text{trip}} \times \dfrac{\text{gal}}{\text{mi}} = 6.\overline{6}\,\dfrac{\text{gal}}{\text{trip}}$$

5. The 1000 seats in an auditorium are placed on sale for a band concert and a large crowd is expected, so a limit of 4 tickets per person is allowed. If each person in a ticket line buys the maximum allotment, how many people will be able to purchase tickets?

Answer: $\dfrac{1000 \text{ tickets}}{4\,\dfrac{\text{tickets}}{\text{person}}} = 250 \text{ people.}$

Comment: Thinking of taking 4 tickets away for each person, this problem has an interpretation as repeated subtraction.

6. The goal of many major league baseball players is to have 3000 hits. If a player has a batting average of .285 (a reasonable lifetime average these days for a good player), about how many times at bat are needed to reach this goal?

Answer: $.285 = \frac{\text{hits}}{\text{times at bat}}$, so the goal is

$$\frac{3000 \text{ hits}}{.285 \frac{\text{hits}}{\text{times at bat}}} \approx 10500 \text{ times at bat.}$$

Comment: This is a formidable number of opportunities to hit. A typical player is a bat officially only 500 to 600 times in a 162-game season. So it might take

$$\frac{10500 \text{ at bats}}{550 \frac{\text{at bats}}{\text{season}}} \approx 19 \text{ seasons}$$

to garner this many hits. Few players play this long. A higher batting average is usually needed to reach the goal.

Division Use Class D: Size Change Divisor

A stuffed animal is twice actual size. Then a leg 30 cm on the stuffed animal corresponds to what length on the real animal? The answer is easily found by division.

$$\frac{30\text{ cm}}{2} = 15\text{ cm}.$$

With 42% of precincts reporting, a candidate has 79,322 votes. Assuming the count continues in about this way, estimate how many votes the candidate will ultimately receive. Again division gets the answer.

$$\frac{79{,}322\text{ votes}}{42\%} \approx 188{,}862\text{ votes}.$$

We would round to 190,000.

These situations are of the form: Given a part (or a multiple) of a quantity, find the quantity. Many people think of these problems as asking for a unit or whole: If 79,322 votes is 42% of the total, what is 100% of the total? If 30 cm is twice actual size, what is one time actual size? A proportion can be used:

$$\frac{30\text{ cm}}{2} = \frac{x}{1} \qquad \frac{79{,}322\text{ votes}}{42\%} = \frac{T}{100\%},$$

but experienced users of arithmetic do the division directly.

These problems are also clearly related to size change multiplication. Specifically, some would set up the first situation above as $2x = 30$ cm and the second as $42\%T = 79{,}322$. The divisor in the use class then becomes the size change factor in the multiplication.

Examples:

1. A salesperson and a car buyer agree on a total price (including tax) of \$7300 for a car in a state with a 5% sales tax. How much of this \$7300 is the price of the car and how much is tax?

   Answer: The 5% tax rate indicates a size change factor of 1.05.

   $$\frac{\$7300}{1.05} = \$6952.38\text{, the price of the car.}$$

   The rest, \$347.62, is to be paid to the state as sales tax.

   Comment: Many people might answer the question by solving the equation $1.05x = \$7300$. The setup comes from size change multiplication, but one ultimately performs the same division as above.

2. In 1976 only 43% of the adults of voting age in Georgia voted in the presidential elections. If a polling organization wants to sample 1000 people who did vote, approximately how many people who were of voting age then will they have to question?

   Answer: $\frac{1000 \text{ people}}{.43} = 2326$ people. One would expect to sample about 2400 people.

   Comment: Many people see this as a multiplication problem

   $$1000 = .43 \times (\text{number of voting adults}).$$

   Comment: One reason so few may have voted is that this was Jimmy Carter's home state and he seemed like a sure winner in that state.

3. A stuffed animal is $\frac{2}{3}$ actual size. Then an arm 30 cm long on the stuffed animal corresponds to what length in the real animal?

   Answer: This problem is identical in form to the example that started this section.

   $$\frac{30 \text{ cm}}{\frac{2}{3}} = 45 \text{ cm, the length desired.}$$

Comment: Many people would not answer the question by a single division, as we did, being distracted by the 2/3. (Most would divide if the 2/3 were replaced by a whole number.) Some would divide by 2, getting 14 cm which is 1/3 actual size, then multiply by 3. Others would multiply by 3, then divide by 2.

Comment: Algebraically, 2/3 x length desired = 30 cm, so length desired $= \dfrac{30\text{ cm}}{\frac{2}{3}}$.

4. After a discount of 30%, the sale price of an item is \$21.70. What was the original price?

Answer: A discount of 30% means that the given price is 70% of the original. The original price can be found by dividing the final price by .70.

$$\frac{\$21.70}{.70} = \$31.00\text{, the original price.}$$

5. The Earth has a diameter about four times that of the moon. If the diameter of the Earth is about 7840 miles, what is the approximate diameter of the moon?

Answer: Divide by the size change factor, 4. $\frac{7840\text{ mi}}{4} = 1960$ mi. The diameter of the moon is approximately 2000 miles.

Comment: This question involves such an easy number, 4, that most people do not have to conceptualize the question as division to be able to answer it.

6. If an ant is 200 times smaller than a human, estimate the length of the ant.

Answer: If the length of most humans is taken between 150 and 190 cm, then the length of the ant is between $\frac{150}{200}$ and $\frac{190}{200}$ cm, that is, between .75 and .95 cm, approximately.

Comment: We don't encourage the use of the terminology "200 times smaller than", perferring "$\frac{1}{200}$ of" but the wording of this example is occasionally found.

7. Probability. In rolling two dice, a sum of 3 is three times as unlikely as a sum of 7. If the probability of a 7 is $\frac{1}{6}$, what is the probability of a 3?

Answer: Divide $\frac{1}{6}$ by 3, yielding $\frac{1}{18}$.

Comment: One can analyze this problem as follows: "Three times as unlikely" means "one-third as likely". That signifies multiplicator of $\frac{1}{6}$ by $\frac{1}{3}$, which is equivalent to dividing $\frac{1}{6}$ by 3.

Division Use Class E: Recovering Factor (In Acting Across)

A typical school book (including teacher manuals, tests, etc.) takes about 70 editor-months to publish, starting from author manuscript. (1) If a company has 5 editors to put on the book, how long will the editing take? (2) If the book has to be edited in a year, how many editors are needed? Each question can be answered with a single division.

(1) $\frac{\text{70 editor-months}}{\text{5 editors}} = \text{14 months}$

(2) $\frac{\text{70 editor-months}}{\text{12 months}} \approx \text{6 editors}$

This illustrates how acting across multiplications can lead to problems requiring division. The compound unit and one of the factors must be given, with the desire to find the other factor.

Examples:

1. Energy use. A monthly electric bill for a large house indicates that 806 kwh were used from Oct. 22, 1980 to Nov. 22, 1980. On the average, how many watts were being used at a given time in this month?

   Answer: 806 kwh = 806,000 watt-hours. To find the desired number of watts, divide by the number of hours. In 31 days, there are 744 hours.

   $$\frac{\text{806,000 watt-hours}}{\text{744 hours}} \approx \text{1083 watts}$$

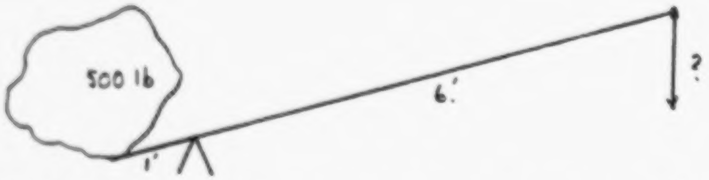
   Comment: Assuming that for 8 hours a day (while the household is asleep), little energy is consumed, one might wish to divide 806 kwh by 496 hours. This gives 1625 watts being used on the average, a very high consumption rate.

2. Depth. A store requires 200 m^2 of space and a vacant lot has 10 m frontage. How deep will the store have to be built?

Answer: If the store is built to the sides of the lot, using all of the width, it will have to be $\frac{200\ m^2}{10\ m}$ or 20 m deep.

Comment: This problem might be classified as "length given area".

3. Levers. A 7-foot lever is to be used to move a 500-lb rock. The fulcrum of the lever is 1 foot away from the rock. How much force needs to be exerted to move the rock?



Answer: $\frac{500\ \text{lb} \times 1\ \text{ft}}{6\ \text{ft}} = 83\frac{1}{3}$ lb, so a force of at least $83\frac{1}{3}$ lb is needed.

Comment: The compound unit here is foot-pounds, a unit of work.

4. What size fuse (in amps) is needed to support a 2000-watt kitchen load (e.g., a toaster and blender operating simultaneously)? Assume standard 120-volt current.

Answer: The formula to use is not $P = E \times I$, but the equivalent

$$I = \frac{P}{E}$$

$$I = \frac{2000\ \text{watts}}{120\ \text{volts}} = 17\ \text{amps, so a 20-amp fuse is needed.}$$

Comment: During peak periods called "brownouts", 120 volts may not be supplied by an electric company. That is, if 105 volts are supplied in the above situation, then about 19 amps are needed. The 20-amp fuse wouldn't blow but the extra current heats up wires and motors and can cause damage to the appliances.

Summary

The two use meanings of division are ratio and rate. Each of these can be conceived of as a comparison, the former comparing quantities with the same unit, the latter comparing quantities with different units. The resulting quotients are, however, quite different. In the case of ratio, the quotient is a scalar, and reversing the quantities being compared leads to its reciprocal. In the case of rate, the quotient is a quantity (with a derived unit), and reversing divisor and dividend results in an equivalent rate, one with both reciprocal units and labels.

Division possesses three other use classes; each related to a use class of multiplication.

rate divisor ⟷ rate factor
size change divisor ⟷ size change
recovering factor ⟷ acting across

The relationship is generally through the related facts: a x b = c if and only if a = c/b or b = c/a. Yet rate factor multiplication can be thought of as being derived from division rate, and size change multiplication from division ratio, so the origin of two of these division use classes may be division itself.

Schoolbooks generally conceive of ratio as apart from division and ignore rate. Only "splitting-up" uses, a small subclass of rate or rate divisor problems, typically are in the student's books. It is thus no wonder that students find applying division to be rather difficult.

Pedagogical Remarks

Getting started. As with any of the operations, one should begin with examples where the numbers are so easy that students can answer in their heads. For example, here is a ratio problem.

> It took Jane 3 minutes to do a problem.
> It took John 12 minutes to do the same
> problem. How many times longer did it
> take John than Jane?

The student usually answers 4 even without dividing. Now ask: What operation could you have done to get the answer? Then ask: Suppose the numbers were changed. Would the same operation work? Then change the numbers. Depending upon the sophistication of the students, you might first change 12 minutes to 15 minutes, or change 12 minutes to 13 minutes, or change 3 minutes to $3\frac{1}{2}$ minutes, etc. Then change the situation. John has $3, Jane $12. How many times more money does Jane have than John. And so on, until the general notion of using division sinks in.

Ratio. When ratios are quotients in division problems, they are to be treated as single numbers. The answer to the above problem is 4, not 4:1.

Ratios can usually be taken in either order. Above, one could ask how many times longer it took Jane than John. Answer: It took Jane $\frac{3 \text{ min}}{12 \text{ min}}$ or $\frac{1}{4}$ as long. Keeping the units seems to help students.

Ratios are numbers used as comparisons. Emphasize that one can compare either by subtraction or division. The former is often used when the two numbers being compared are close to one another in size. The latter is more often used when they are far apart.

Fractions and division. For uses of division, it is essential that a student be able to interpret $\frac{a}{b}$ as "a divided by b", and as a single number, the quotient of a and b. The reason for this is that in most applications of division, the problem is set up using fraction notation.

Rate. The "splitting up" problems found in many books are quite suitable as first examples of rates. For example:

> 50 cookies are eaten by 10 guests at a party.
> How many cookies did each guest eat, on the average?

Keep the units in the divisor, dividend, and quotient, and emphasize the the phrase "cookies per guest" indicates which units go where. As with the example given on the previous page, after students realize that division can be used to get the answer, change the numbers so that the answer does not come out even.

> 48 cookies are eaten by 9 guests at a party...

The answer $5\frac{3}{9}$ indicates that all guests can get $5\frac{1}{3}$ cookies. The answer "5 with remainder 3" indicates that all guests can get 5 cookies but there will be 3 left over. Either answer can be checked by multiplication.

$$\text{Check 1: } 9 \text{ guests} \times 5\frac{1}{3}\,\frac{\text{cookies}}{\text{guest}} = \ldots$$

$$\text{Check 2: } 9 \text{ guests} \times 5\,\frac{\text{cookies}}{\text{guest}} + 3 \text{ cookies} = \ldots$$

Notice that the check also verifies the unit of the answer.

If students have studied fractions or decimals, make certain that examples of rates are given which involve measures that are not whole numbers, so that they can see real-world uses for division of rational numbers. Among rates which involve measures are speed (mi/hr, km/hr, km/sec, etc.), pressure (lb/sq in, gm/cm^2, etc.), weight gain (lb/week, kg/month, etc.), and density (mass/volume). A rate in which a measure is in the denominator allows one to set up problems in which the divisor is a fraction or decimal.

Rate divisor. Though these problems, when set up with measure labels included, are quite complicated, research indicates that they may be among the easiest of division problems for students to understand. Many books begin division with rate divisor uses.

50 cookies are to be split among guests
so that each guest receives 5 cookies.
How many guests can be served?

$$\text{Answer: } \frac{50 \text{ cookies}}{5 \dfrac{\text{cookies}}{\text{guest}}} = 10 \text{ guests.}$$

The unit arithmetic here follows the standard rules for division of fractions:

$$\frac{\text{cookies}}{\dfrac{\text{cookies}}{\text{guest}}} = \text{cookies} \times \frac{\text{guests}}{\text{cookie}} = \text{guests}$$

The reason for the ease with which these problems are learned many be that one ignores the label "guests" and does the problem as a ratio problem. That idea is easy to apply in many rate divisor situations, and we do not discourage it. The teacher may wish to do problems both ways, so as better to prepare students for work with labels in fractions.

Size change divisor. First examples should use simple numbers.

(1) A picture of an insect shown in a book is 3 times actual size. If the length of the insect in the picture is 6 cm, how long is the actual insect?

In such situations, a size change factor and the result of a size change are known and one wishes to determine the quantity before the size change. As with the other use classes for division, after the idea has been learned with simple numbers, it is important to change the numbers (but not necessarily the situation).

(2) A picture in a book is 14 times actual size. If a length in the picture is 2.3 cm, how long is the actual length?

(3) A picture in a book is $\frac{1}{4}$ actual size. If a length in the picture is 8 cm, how long is the actual length?

To answer (3), you could multiply: 8 cm x 4 = 32 cm. However, you could also divide: $\frac{8\text{ cm}}{\frac{1}{4}} = 32$ cm. The change from division to multiplication provides a nice example of the "invert and multiply" rule found with division of fractions. That is, to teach this rule, you might want to give a division size change problem where the size change factor is a unit fraction.

What are often reported to be the most difficult of percent problems fall into this class. Using the same context as above:

(4) A picture in a book is 23% actual size. If a length in the picture is 5 cm, how long is the actual length?

Answer: $\frac{5\text{ cm}}{.23} \approx 22$ cm.

For students who have trouble with this use class, the relationship with multiplication should be exploited. In (4), that would be:

23% x actual size = 5 cm

But still, one must divide 5 cm by .23 to get the answer, so there is benefit in thinking of division from the beginning. That can be done by thinking as follows: If 5 cm is 23% actual size, what is (100%) actual size? Then set up a proportion

$$\frac{5\text{ cm}}{23\%} = \frac{x\text{ cm}}{100\%}$$

and, since 100% = 1, the same division gets the answer.

Recovering factor. Since every division problem $a \div b = c$ is equivalent to a multiplication problem $b \times c = a$, in a theoretical sense every division problem could be termed "recovering factor". In this volume, however, we have reserved this use class for those uses of division that arise from the acting across uses of multiplication--i.e., those uses in which the unit of the product is a compound of the units of the factors. For example, kilowatt-hours from kilowatts and hours, couples from boys and girls, cm^2 from cm and cm, and so on.

One way to introduce this use class is with multiplication situations and, by interchanging what is given and what is to be found, create a division situation. For instance:

Multiplication:	If a 100-watt bulb burns for 3 days (72 hours), how much energy is used? (Answer: 7200 watt-hours, or 7.2 kwh.)
Division:	If 7.2 kwh hours have been used in 3 days, how many watts are being used at a given time?

In solving the division problem, keep the units in mind. Then it's easy.

$$\frac{7.2\text{ kilowatt-hours}}{72\text{ hours}} = .1\text{ kilowatts } (= 100\text{ watts})$$

Questions

1. What percent is 3 students in a class of 20?
2. At one fill-up, a car gets 12.3 gallons of gas. 300 miles later the car's tank needs 11.4 gallons to be filled. How many miles per gallon is this?
3. How are the three labels related: words; words per minute; minutes?
4. What rate, with the unit $\frac{hr}{mi}$, is equivalent to 30 mph? Use your answer to convert 30 mph to minutes per mile.
5. Calculate two equivalent rates with different units from the following data: Harold lost 4 kg in 20 days.
6. Make up three problems, one with each piece of data missing, and classify the problems by use class. Use the following information. A scale drawing upon which 2 ft in the world is 3 inches in the drawing is $\frac{1}{8}$ actual size.
7. Make up two division problems from the following situation. A school has 3 6th grade classes with an average of 27 students in each, for a total of 81 students.
8. After an announced 20% increase in fares, you take a taxicab ride and pay $7.20. What would the fare have been before the increase?
9. It is possible to have a shelf with top area of 1 sq ft and a length of 2 ft? If so, what will be the dimensions of the shelf? If not, why not?
10. A person can do a third of a job in an hour. At this rate, how long will it take to do half the job? (Hint: use the units as hints to what numbers to divide.)

11. A school has 2430 students one year and 2296 the next. What is the percent decrease?

Notes and Commentary

1. The traditional classification for uses of division
2. Sutherland's classification
3. Zweng's classification
4. Work of others
5. Our previous work
6. Repeated subtraction as a use meaning
7. Other use classes for division
8. Uses and meanings of fractions
9. Resolving the confusion regarding ratios
10. Percents as ratios
11. Fractions
12. Two types of comparison
13. Rate vs. ratio

1. *The traditional classification for uses of division*. Many methods books for elementary school teachers speak of two types of division problems and no others. The first type is exemplified by the following problem, taken from Grossnickle and Reckzeh (1973).

> "How many 8¢ postage stamps can be purchased for 40¢?"

In this type, given is a total measure and a partial measure, and one must find the number of times the partial measure goes into the total. An equivalent problem is, given the count of set and the size of equal-sized disjoint subsets whose union is the set, to find the number of subsets. Various names have been given to this use type: *ratio*, *comparison*, *measurement*, *quotitive division*.

We would call this first type rate divisor.

$$\frac{40\text{¢}}{8\ \frac{\text{¢}}{\text{stamp}}} = 5 \text{ stamps}$$

The second type is variously labelled *rate* or *partition* or *partitive* division and is exemplified by the following problem, also from Grossnickle and Reckzeh (1973).

> "Tom bought 8 candy bars (each costing the same) for 40¢. What is the cost of one candy bar?

In this instance, a total measure and number of partitions is given, and one has to find the partial measure. Or, in set language one is given the count of a set and the number of equal-sized subsets, and one must find the count of each subset. We call this type *rate*.

Thus our classification differs in language from the common usage. We identify more use classes than most authors, and are able to classify some problems that others cannot.

2. Sutherland's classification. We complete our coverage of Ethel Sutherland's classification (1947). She first remarks "Every division problem represents one of two fundamental meanings--measurement or partition". Then she details eight patterns for each, split into six categories, as follows:

Patterns Representing the Measurement Concept of Division

I. Simpler patterns requiring a number of groups to be formed
  1. Given a total number of like units and the number of units in each group, find the number of groups that can be formed.
  2. Given the total cost or amount and the cost or amount of one unit, find the number of units purchased or required. The word amount, as used in this pattern, refers wholly to money.
  3. Given the total amount to be paid, saved, or done and the rate of doing it, find the total time needed.

II. The patterns involving the concepts of distance, rate, and time
  4. Given the total distance and the rate per unit per time, find the time.
  5. Given the total distance and the distance covered per gallon, find the number of gallons.
  6. Given the total time and the time per unit of of distance, find the total distance.

III. The patterns involving the comparison of two numbers
  7. Given two numbers, how many times as large is one number as the other?
  8. Given two numbers, the smaller number is what part of the larger number?

Patterns Representing the Partition Concept of Division

IV. The patterns that require the finding of the amount, the size, or the cost of each part
  9. Given the total number, find a fractional part of it, the numerator of the fraction being one. (This is considered as division in grades 3 and 4, multiplication in grades 5 and 6.)
  10. Given the total amount and the number of parts into which it is to be equally divided,find the number or size per part.
  11. Given the total amount and the number of persons involved, find the equal share of each person.
  12. Given the total cost and the number of like articles purchased, find the cost of one article.

BEST COPY AVAILABLE

V. The pattern involving the concept of an average (but not involving the concept of distance)
   13. Given the total amount and the number of units involved, find the average per unit.

VI. The patterns involving concepts of average in relation to distance, rate, and time
   14. Given the total distance and the time, find the rate per unit of time.
   15. Given the total distance and the number of gallons, find the distance per gallon.
   16. Given the total time and the total distance covered, find the time required to cover one unit of distance.

She found patterns 5, 6, 15, and 16 to appear rarely in books, while all others were quite well represented. Her analysis covered all the grades 3-6, yet she found no examples of what we would call size change divisor or recovering factor uses. The total absence of size change divisor uses in the curriculum may account for the difficulty later students have with percentage and decimal prolems involving this type of situation.

3. Zweng's classification. Zweng (1963) separated measurement and partitive situations into two types each. Here are her examples for these types.

> "Basic Measurement If I have 8 pencils and separate them into sets of two pencils, how many sets will I obtain?
> Rate Measurement If I have 8 pencils and put the pencils into boxes, placing two pencils in each box, how many boxes will be used?
> Basic Partitive If I have 8 pencils and separate them into 4 sets with the same number of pencils in each set, how many pencils will there be in a set?
> Rate Partitive If I have 8 pencils and put them in four boxes with the same number of pencils in each box, how many pencils will there be in each box?" (pp. 12-13)

To us there is nothing but a semantic difference between the basic and the rate types (i.e., sets vs. boxes). So as a classification we find this weak. However, Zweng found that the rate measurement and rate partitive problems were easier than the basic measurement and basic partitive problems, with differences particularly substantial in the partitive case. (Her subjects were second-graders and had not studied division in school.) She argues that the introduction of the notion of set or group of objects is quite abstract, and the second concrete unit (here "boxes") is a help rather than a hindrance. We agree and note the implication that our structurally more complex rate divisor use class is perhaps easier to understand in some instances than our structurally simpler ratio use class. We also note that Zweng's use of "rate" is different from most other authors on the subject and differs from our use.

BEST COPY AVAILABLE

4. Work of others. Kennedy (1970) identifies three division situations: measurement, partitive, and ratio. The last is applied to problems of the type "How many times greater...?" Thus he noticed, as we did, that earlier classifications tended to neglect situations in which the quotient is a scalar, what we have called ratio.

5. Our previous work. Usiskin (1976) has four uses of division: splitting up (covering the discrete counting instances of partitions of sets), rate, ratio, and repeated subtraction.

6. Repeated subtraction as a use meaning. The long division algorithm is based upon treating division as repeated subtraction. Some real world problems fit this conception. For example:

> If 5 sheets of paper are used up every day, how long will it take to use up a ream (500 sheets) of paper?

The repeated subtraction arises from thinking of taking away the 5 sheets each day. Since the 5 sheets can be repeatedly taken away only 100 times, the quotient (found by repeated subtraction) is 100 days. We did not classify repeated subtraction as a separate use meaning because this and all other problems of the repeated subtraction type easily fit an already-existing use class. E.g., this particular problem fits the rate divisor use class:

$$\frac{500 \text{ sheets}}{5 \frac{\text{sheets}}{\text{day}}} = 100 \text{ days.}$$

We believe that one of the reasons students have so much difficulty with the long division algorithm is that the repeated subtraction upon which it is based does not lend itself to easy real-world analogies. Thus the student cannot easily rely upon concrete materials or real world situations to picture the algorithm. This is in the contrast to the notions of borrowing and carrying in the usual subtraction and addition algorithms, for which real-world counterparts are rather evident.

7. Other use classes for division. An early draft of this handbook included conversion rate as a use class, with problems where the quotient is a conversion rate.

> There are 2.54 cm in an inch. To convert centimeters to inches, by what should one multiply?

The answer to such a problem--the conversion rate 1 in/2.54 cm--possesses properties that identify it simultaneously as both a rate and a ratio. On the one hand, the quantity 1/2.54 in/cm looks like a rate. We even say "inches per centimeter". On the othr hand, the numerator and denominator of 1 inch/2.54 cm are identical, so that this fraction equals the number 1, and thus is a scalar (as are all ratios).

BEST COPY AVAILABLE

We opted for classifying conversion rates under rates because of their structural form and because this choice is suggested by popular language. We think the other possibilities, classifying conversion rates under ratios or identifying another use class (but not another use meaning) are reasonable as well.

8. Uses and meanings of fractions. Because one of the ways of thinking of fractions is as an indicated division, uses of fractions are candidates for uses of division. Analyses of the uses of fractions are not helped by inconsistent language in current use. Here are some examples. (1) The recently published yearly indexes of the Arithmetic Teacher, under the heading "Decimals", have no listings, but instead refer the reader to "Fractions". This seems to assume that all decimals are fractions, yet the most obvious meanings of "decimal" and "fraction" are as distinct notations for numbers. (2) Kieren (1976, 1980), has written extensively about rational numbers. Yet his rational numbers are exclusively fractions with whole numbers in numerator and denominator. This suggests the mathematically incorrect notion that "rational number" and "fraction" are synonyms. (3) The phrase "decimal fractions" usually refers to 3.5, 4.23, .000333..., and other numbers written with digits to the right of the decimal point. Thus, unlike the fraction a/b, which has a numerator and denominator, decimal fractions never have these. (4) In common parlance, we often say that the answer is "a fraction" when we mean that the answer is not a whole number, as in "There are no problems in that set in which the answer is a fraction". Under this usage, one might wrongly deduce that 4/1 is not a fraction.

The point of this is not necessarily to be critical. One cannot legislate usage. However, we conclude that the word "fraction" is so variously and inconsistently employed that one has to be quite wary when examining lists of uses of fractions. Acordingly,, we find that others' lists of uses of fractions correspond to uses found in diverse places in this volume nd sometimes to mathematical meanings not considered by us to be uses at all.

For example, Swenson (1973) identifies four uses of fractions: fractions to represent parts of units, fractions to represent parts of sets, fractions to express ratios, and fractions to indicate division. These correspond respectively to one of the measurement uses of numbers, a special case (with counts) of the ratio use of numbers, the ordered pair representation for ratios, and the mathematical definition as quotients. So Swenson's uses represent single number uses, n-tuple uses, and a mathematical meaning.

Kieren (1976) gave six conceptions of rational numbers: as equivalence classes of fractions, ratio numbers, operators or mappings, elements of a quotient field, measures, and decimal fractions. Later (Kieren, 1980) he reduced these to four, perhaps to reach a less sophisticated audience: measure numbers, quotients, ratio numbers, and operators. These last four correspond to

BEST COPY AVAILABLE

rational numbers derived as measurements (a use of numbers), quotients (a mathematical definition as a single number), ratios (a mathematical definition as an ordered pair), and as size change factors (as in the size change use class of multiplication). Thus Kieren's categories represent single number uses, operation uses, mathematical meanings and use meanings.

9. *Resolving the confusion regarding ratios*. We hope that our analysis of the uses of numbers and operations has not only identified problems others have had in analyzing ratios, fractions, and division, but has also provided a solution. Here is a summary of our views.

(A) One of two mathematical formulations of the concept commonly called *ratio* is as an ordered pair. Accordingly, ratios have some of the uses of ordered pairs. They are used for comparison (2 out of 3 people in the office are absent). When we "add" ratios as ordered pairs, the "sum" is found by adding corresponding components. (If 4 of 5 are absent in a second office then we add corresponding components to find that 6 of 8 are absent in the two offices.) Ratios as ordered pairs can be put in order (4 of 5 is a greater ratio than 6 of 8). There are equivalence classes of these ratios (2 of 3 is equivalent to 4 of 6). But the "addition" (quotes necessary, for this is not addition of real numbers) does not preserve order or equivalence classes.

(B) The second mathematical meaning to the concept commonly called ratio is as a quotient, a single number (just as a sum or product is a single number). For us, and some in science (e.g., Goodstein (1982)), the divisor and dividend used in forming this ratio must have the same unit label, so that the quotient--the ratio itself--represents the use of number called ratio comparison. For some others, the divisor and dividend may be any counting or measure units, so the quotient may be a rate and represent a measure rather than a ratio comparison.

Whereas ratios as ordered pairs are subject to operations on ordered pairs, ratios as single numbers follow the ordinary operations of arithmetic. Thus ratio comparisons may under some cirumstances be meaningfully added, subtracted, multiplied, divided or used in powering. Since ratios as single numbers are (in our characterization) scalars, when they are used in these operations, they are used only in ways that scalars are used. Thus, for example, one may have a ratio in multiplication size change but not in multiplication acting through.

(C) When ratios are being used as ordered pairs, it is natural to want to convert them to single numbers. The ordered pair (a,b) is converted to the single number quotient a/b. This conversion loses the original data, the ordered pair, but one can, with the quotient, apply all of the operations of arithmetic as appropriate.

10. *Percents as ratios*. All ways in which the word "ratio"

BEST COPY AVAILABLE

is used apply to percents. We think of a percent as a single number: 50% = .5 = 1/2. We think of a percent as an ordered pair: 50% means "50 out of 100". And we think of a percent as involving division: 50% means "50 divided by 100". So the same confusions surrounding ratio are associated with percents.

11. Fractions. Fractions are often introduced in ways that ignore both their capacity to represent a single number and their link to division. Specifically, a student is told what the numerator and denominator represent, but not what the fraction represents. Yet the single entity conceptions are the important conceptions one must have to be able to apply fractions. To us, a fraction is a form $\frac{a}{b}$ or a/b, where a and b are either scalars, quantities, or--as in unit arithmetic--labels. If numbers are involved, a fraction can always be interpreted as a quotient (i.e., as a single number or quantity) but the division need not have taken place. (For example, the number 1/2 can be said to exist apart from any operation of division.)

A fraction may represent an irrational number, such as /2, 2/3. A fraction may represent a whole number, such as 4/1 or 6/6. The numerator and denominator of fractions can be fractions or decimals, as in (3/5)/(1/2) or 2.5/1.6. The popular usage of the word "fraction" as denoting a "rational number that is not a whole number" is too restrictive and mathematically incorrect.

12. Two types of comparison. Just as ratio has more than one mathematical referent, so too the word comparison is used in two ways. There are many analogies between subtraction comparison and ratio (division) comparison. Suppose a and b are quantities to be compared.

Subtraction Comparison	Division Ratio
1. a and b have the same labels then a-b makes sense and has that label.	1. If a and b have the same labels, and b is not 0, a/b makes sense as a ratio.
2. Switching order of comparison yields answers that are opposites.	2. Switching order of comparison yields answers that are reciprocals.
3. Two equal quantities yield a difference of 0.	3. Two equal quantities yield a ratio of 1.
4. One can compare a to c by adding results of comparing a to b and b to c. That is, $(a-b) + (b-c) = a-c$	4. One can compare a to b by multiplying results of comparing a to b and b to c. That is, $\frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c}$

13. Rate vs. ratio. Our distinction between the two division comparisons ratio (same label) and rate (different

BEST COPY AVAILABLE

labels) is that made by scientists [e.g., see Goodstein, (1982). Independently, we have found it to be most useful (Usiskin, 1976) both pedagogically and in problem-solving.

However, the distinction is not always easy to make and is not always made in the real world. We have already mentioned that conversion rates share properties of both rates and ratios, and thus have somewhat of a dual existence in our schema. The unemployment "rate" $\frac{\text{no. of jobless}}{\text{no. in work force}}$ could be considered either as a rate (with the unit being jobless per work force) or as a ratio (people to people). We prefer to think of the unemployment rate as a ratio because the units do not come into play in discussions, but there are some calculations in which consideration as a rate would be advised. For example, if 75% of the jobless were eligible for benefits, then the ratio of the workforce receiving unemployment benefits is

$$\frac{\text{no. with benefits}}{\text{no. ofjobless}} \times \frac{\text{no. of jobless}}{\text{no. in work force}}$$

$$= 75\% \times 8\%$$

$$= 6\%$$

and the rate units clarify the computation.

BEST COPY AVAILABLE

Summary of the Use Classes of the Four Fundamental Operations*

			Addition	Subtraction	
a	b	c	a+b=c or b+a=c	c−b=a	c−a=b
part	part	whole	putting together	take-away	
input	shift	output	shift	subtraction shift	comparison

			Multiplication	Division	
a	b	c	a×b=c or b×a=c	c÷b=a	c÷a=b
size change factor	input	output	size change	ratio	size change divisor
quantity unit 1	quantity unit 2	quantity unit 1-unit 2	acting across	recovering factor	
rate $\frac{\text{unit 2}}{\text{unit 1}}$	quantity unit 1	quantity unit 2	rate factor	rate	rate divisor

---

* Many instances of uses of the operations can be interpreted in more than one way.

BEST COPY AVAILABLE

CHAPTER 9

USES OF POWERING

We use the word powering to describe the operation inherent in expressions of the form 2^3 or 10^{-5}. Most students do not deal with powering until the sixth grade and certain aspects of powering are often first discussed in second-year algebra, so this operation and its terminology are not likely to be as familiar to readers as the operations of previous chapters. Still, the uses of powering are widespread and significant.

Mathematical background. Because of the relative unfamiliarity of the operation of powering, we give here a brief introduction to the language and properties associated with this operation. Even the reader familiar with this operation should skim this discussion.

In the expression x^y, x is the base and y is the exponent. The expression x^y is the yth power of x. That is, the result of powering is called a power. The second power of x, x^2, is called x squared; the third power of x, x^3, is called x cubed. In this chapter, we restrict applications to those in which the base x is a positive number.

For any positive number x, the zero power $x^0 = 1$. The first power of x, $x^1 = x$. Powers with small whole number exponents can be dealt with by multiplication: $x^2 = x \cdot x$, $x^3 = x \cdot x \cdot x$, etc. For example:

$$4^0 = 1$$

$$4^1 = 4$$

$$4^2 = 16$$

$$4^3 = 64$$

and each succeeding whole number power is 4 times the previous one.

It takes two numbers to get an answer in a powering situation, so powering is a binary operation, a property it shares with the other operations we have discussed. Switching base and exponent makes a difference; for example, $2^3 = 8$ but $3^2 = 9$. So powering is not commutative.

Until the advent of calculators, it was difficult to calculate or estimate decimal approximations for anything but small whole number powers of most numbers. Today, with even the simplest calculator, one can quickly find or estimate x^n when n is a whole number and, if the calculator has an x^y key, all powers can be quickly estimated by decimals.

Powers with negative exponents are reciprocals of corresponding powers with positive exponents.

$$4^{-1} = \frac{1}{4}$$

$$4^{-2} = \frac{1}{16}$$

$$4^{-3} = \frac{1}{64}$$

and, in general, $$x^{-n} = \frac{1}{x^n} = \text{the reciprocal of } x^n$$

Certain powers with non-integer exponents are related to roots of numbers, and can be written using the radical sign $\sqrt{\ }$:

$$x^{1/2} = \text{the square root of } x = \sqrt{x}$$

$$x^{1/3} = \text{the cube root of } x = \sqrt[3]{x}$$

and, in general, $$x^{1/n} = \text{the nth root of } x = \sqrt[n]{x}$$

$$x^{m/n} = \text{the nth root of } x^m = \sqrt[n]{x^m}$$

$$= \text{the mth power of } x^{1/n} = (\sqrt[n]{x})^m$$

For example, $64^{1/2} = 8$, $64^{1/3} = 4$; $64^{2/3} = 16$. Most non-integer exponents do not yield whole number answers.

The simplest application of powering is when two numbers are so related that one is a power of the other. For example, if z is the yth power of x, then

$$z = x^y.$$

In this case, it also is the case that x is a power of z, for we may take the yth root of each side to achieve

$$z^{1/y} = x.$$

For example, since $4^3 = 64$, it is also the case that $64^{1/3} = 4$. One consequence of this relationship with regard to applications is that for any application that involves a whole number exponent, there is a potential application involving an exponent that is a unit fraction, hence an application involving roots. For example, consider the volume V and length of a side s of a cube. The volume formula is

$$V = s^3$$

and thus

$$V^{1/3} = s.$$

Many applications of powering involve situations in which one number z is a multiple of a power of a second number x. That is,

$$z = kx^y$$

Here x is raised to the yth power, but k is not. By multiplying both sides of this formula by x^{-y}, one solves the formula for k.

$$zx^{-y} = k$$

So for any application of this type that involves a positive exponent, there is a potential application involving a negative exponent.

The point we wish to emphasize here is that the uses of powering involve more than whole number exponents. In keeping with our policy of not distinguishing uses by the particular numbers that happen to be involved, the examples in the sections of this chapter include uses of powering with all kinds of exponents: zero, negative, and non-integer. Yet the reader who is unfamiliar with negative or non-integer exponents should not be dismayed. All of the use classes of powering can be exemplified with whole number exponents and the simplest calculators can handle these.

Powering has two use meanings:

A. Change of dimension
B. Growth

and one use class derived from its relationship with multiplication,

C. Notation.

Powering Use Class A: Change of Dimension

The simplest examples of change of dimension are from side-length to area in a square, and from side-length to volume in a cube. It is from these notions that the names "x squared" for x^2 and "x cubed" for x^3 arose.

side length x

side length x

The labels associated with area and volume also utilize powering notation. For instance, when a side is measured in cm (or ft), the corresponding area must be in cm^2 (or ft^2) in order for area formulas to hold true, and the corresponding volume is in cm^3 (or ft^3). The substitutes sq cm and cu cm (or cc) and the very common sq ft and cu ft are employed so as to avoid the exponent and make it easier to type the label (see Chapter 11, Section D regarding change of notation).

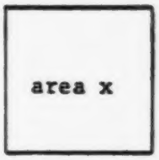
The length (or area or volume) of an object is, in applications, the ratio of the size of the object to the size of a selected unit of length (or unit of area or unit of volume). Uses involving change of dimension all arise from the following general use meaning.

Change of dimension use meaning for powering

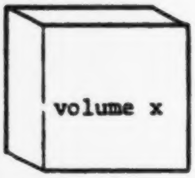
If lengths in two similar objects are in the ratio L, then corresponding quantities of dimension d in these objects are in the ratio L^d.

For area, d = 2; for volume, d = 3; for length, d = 1, but other dimensions are possible (see note 6).

Square roots historically arose from the need to determine the length of a side of a square from its area. Cube roots similarly arose from determing the length of a side of a cube from its volume. These are pictured below. By combining these roots and powers, $x^{2/3}$ is the area of one face of a cube whose volume is x. Equivalently, $x^{3/2}$ is the volume of a cube whose face has area x.



side length $\sqrt{x}$ or $x^{1/2}$



side length $\sqrt[3]{x}$ or $x^{1/3}$

Applications in this use class arise in two basic ways. First, since area is customarily measured in square units and volume in cubic units, formulas connecting area, volume, and length make use of powering. Second, many attributes in the physical and biological world vary with area, volume, or combinations of them. For example, for a particular mammal, heat loss varies with its surface area, and weight varies with its volume. So in calculating needed caloric intake (to counterbalance heat loss) for a given weight, one is essentially changing dimension from volume to area, and consequently the exponent 2/3 is needed.

Examples:

1. One of two similarly shaped precious stones is 2.5 times the length of the other. How will their weights compare?

   Answer: Weight varies with volume. If the ratio of lengths is 2.5, the ratio of volumes is 2.5^3, so the ratio of weights is also 2.5^3 or 15.625.

   Comment: Many people underestimate this ratio.

2. A recipe calls for a souffle to be made in a 10" by 5" dish. A cook has only circular dishes. What diameter circular dish should be used?

   Answer: The recipe indicates area of the dish and we need diameter, a length. So one should expect to need square roots.

   Area of circular dish = Area of 10" by 5" dish.

$$\pi r^2 = 50 \text{ in.}^2$$

$$r^2 = \frac{50}{\pi} \text{ in.}^2$$

$$r = \sqrt{\frac{50}{\pi} \text{ in.}^2} \approx 3.99 \text{ in.}$$

   A dish with radius 4" (diameter 8") should be used.

   Comment: Notice that the $\sqrt{\text{in.}^2}$ is in., as needed.

3. Two pizzas with the same thickness and ingredients have diameters of 12" and 16". Judging from ingredients alone, if the smaller pizza costs $4, what should be the price for the larger one?

   Answer: Because the pizzas have the same thickness, their ingredients are determined by their surface areas, that is, by the squares of corresponding lengths. The larger has lengths 16/12 times the smaller, so its surface area is $(16/12)^2$ times the smaller.

$$(16/12)^2 = (4/3)^2 = 16/9.$$

So, considering only the ingredients, the larger pizza should cost $\frac{16}{9}$ the smaller, or $\frac{16}{9}$ x \$4, or about \$7.11.

Comment: Of course, a pizza parlor should consider more than ingredients in determining cost. There are costs of operating the parlor (heat, light, taxes, etc.) called fixed costs. There are costs of making pizzas that are not related to area (time to take an order, time tables are occupied, etc.,), but are related to the number of pizzas. Hence a better model than a simple powering is a polynomial:

$$ax^2 + bx + c,$$

where ax^2 is based on ingredients, bx is related to the number of pizzas, and c is the fixed costs.

4. Suppose an oven is on at constant temperature. Then the heat put into the food varies according to the length of time the food is cooked. The heat needs vary with the surface area of the food (just as heat loss varies with this area), but one is usually given the weight of food and not its surface area. To change weight W to surface area (and thus heat need and thus time need) T, one can utilize the following formula:

$$T = kW^{2/3}.$$

According to The Joy of Cooking, a 10-12 pound stuffed turkey should take between 3 1/4 and 3 3/4 hours at 350°. Using the lower and upper limits, what two values of k do you get? According to these values of k, about how long should it take for a 16-pound stuffed turkey at this temperature?

Answer: Using 3 1/4 hours and 10 pounds, one must solve

$$3.25 = k\cdot 10^{2/3}$$

$$3.25\cdot 10^{-2/3} = k$$

$$3.25\cdot 2.15 = k$$

$$.70 = k.$$

Using 3 3/4 hours and 12 pounds, similarly solving

$3.75 = k \cdot 12^{2/3}$,

$k = 3.75 \cdot .191 = .72.$

So a 16-pound turkey should take T hours, where

$$T = .71 \cdot 16^{2/3} = 4\ 1/2 \text{ hours.}$$

Comment: The Joy of Cooking recommends 3 3/4 hours to 4 hours for a 14-18 pound turkey. We speculate that the calculated time would be closer to their recommendations if we were working with weights that included the stuffing--a hollow bird no doubt cooks faster than a solid one.

Comment: See also M. K. Klamkin, "On Cooking a Roast", SIAM Review, April 1961, pp. 167-169.

5. The world record for weightlifting by a person weighing 60 kg or less is 298 kg (sum total of the snatch and jerk lifts) by Viktor Mazin of the USSR. The corresponding world record for a person weighing 100 kg or less is 416 kg by Ota Zaremba of Czechoslovakia. Muscle strength is related to cross-sectional area, weight to volume, so strength varies as the 2/3 power of weight. Which of these people is lifting more for his weight?

Answer: The formula relating strength S and weight W is of the form

$$S = kW^{2/3}.$$

The question can be restated as: For which man is the value of k larger?

Substituting for Mazin,

$$298 = k60^{2/3}, \text{ from which } k = 298 \cdot 60^{-2/3} = 19.4.$$

For Zaremba,

$$416 = k100^{2/3}, \text{ from which } k = 4.6 \cdot 100^{-2/3} = 19.3.$$

So Mazin, the lighter lifter, is stronger, but not by much.

Comment: The values are so close to each other that the difference may lie in the actual weights of the men, which are usually close to the 60 kg and 100 kg limits, but may be a kg or two less.

Comment: This analysis compares the strengths of weightlifters in different weight classes. To our knowledge it has never been applied in competition, perhaps because the mathematics is not well-enough known to people in that field. The closeness of the values (in contrast to Question 4 immediately above) indicates how correct this mathematical model is.

6. For similar objects in water, the velocity that can be attained varies as the square root of the length of the object. That is,

$$V = kL^{1/2}$$

Given similar shape and proportional power, how much faster should a ship which is 200 m long be able to go than one which is 100 m long?

Answer: Proportional power means that the value of k is the same. So the question asks to compare $k200^{1/2}$ to $k100^{1/2}$.

$$k200^{1/2} \approx 14.1k$$

$$k100^{1/2} = 10k$$

Since $\frac{14.1\ k}{10\ k} = 1.41$, the longer ship can go about 1.4 times as fast.

Comment: In general, if one ship has length B times a second similarly shaped ship, then the first ship can go B times as fast as the second. This accounts for the longest superliners like the Queen Elizabeth II being the fastest ships on the sea. It also explains why whales are the fastest swimming mammals. (For details, see "On Magnitude", by D'Arcy Wentworth Thompson, in The World of Mathematics, edited by James R. Newman, p. 1008.)

Powering Use Class B: Growth

Suppose a copy machine reduces originals. A 38% reduction is employed to transfer two 8 1/2" x 11" sheets onto one. Suppose such a reduction is done four times. What is the final copy length for an original length of 10"?

To answer this question, note that each copy is 62% of the earlier one. Thus the fourth copy is

$$(.62)^4 = .1477\ldots \text{ or about } 15\%$$

of the original. So an original length of 10" becomes 1.5". This problem exemplifies the growth-decay uses of powers. We call this use class growth.

A typical growth use is found in calculations dealing with population size. In 1975, according to United Nation's statistics, the World's population reached 4 billion and was growing at the rate of 2% a year. At that rate, each year has the previous year's population multiplied by 1.02. So in n years, the population is multiplied by 1.02^n. For example, if this rate continued for 10 years after 1975, the population in 1985 is given by the expression

$$4 \text{ billion} \times (1.02)^{10}.$$

Such a calculation used to be horrendous. Now with calculators, the decimal approximation 4.86 billion is easy to obtain. It is not possible to take censuses each year, so such approximations are often used.

The same idea can be used to go backward in time. Assuming a population of 4 billion in 1975 and a 2% growth rate in the years before this, the population in 1968 (seven years previous) can be estimated by

$$4 \text{ billion} \times (1.02)^{-7}, \text{ or about } 4 \text{ billion} \times .87.$$

Multiplication shows this to be about 3.5 billion.

The same idea can also be used to get populations in the middle of years. For example, if one assumed a population of 4 billion on January 1, 1975, then the population on July 1, 1976 (1.5 years later) is given by

$$4 \text{ billion} \times (1.02)^{1.5} \approx 4 \text{ billion} \times 1.03...$$

or approximately 4,120,000,000.

Notice the role of 1.02 and 1.5 in the last paragraph. The 1.02 is a constant size change or growth factor in a time interval of length one year. The 1.5 is the number of years. These notions generalize to an important use meaning for powering.

Growth use meaning for powering

Let x be a size change factor. If

(1) x is used y times or

(2) x is applied over each unit interval in an interval of length y, then the original quantity is multiplied by x^y.

In the above example, the interval of unit length is a year. The size change factor x is 1.02, and y took the value 1.5.

In growth uses, the size change factor x must be a positive number. However, y can be any real number. When y is negative as with -7 in 7 years ago, the idea is that of undoing the growth. When x is less than one, as in copy reduction, loss or decay is occurring.

Examples:

1. At 8% annual interest, what will $1000 be worth in five years?

Answer: An 8% growth rate corresponds to a size change factor of 1.08. So $1000 will become $\$1000(1.08)^5 \approx \$1469.33$.

Comment: The unit interval here is one year. In ten years, $1000 would become $\$1000(1.08)^{10}$ or about $2158.93, more than doubling.

2. In order to have \$25,000 for their child's college educaton at the end of high school, how much should the parents invest at the beginning of high school if they can get a 10% annual return?

   Answer: Think back from the \$25,000. The yearly change factor is 1.10. Four years earlier they would have had to invest $\$25000(1.10)^{-4}$ or approximately \$17,075.

   Comment: Without calculators neither of examples 1 or 2 would be easily accessible.

3. Some bacteria colonies are known to grow exponentially over short periods of time. They often double in number in 30 minutes. What will happen in 3 hours at this rate of growth?

   Answer: The unit interval here is 30 minutes. There are 6 30-minute periods in 3 hours. Doubling is a size change factor of 2, so the population would be multiplied by 2^6, or 64.

   Comment: In eight hours, the population would be multiplied by 2^{16}, or approximately 65,000.

   Comment: Such a constant growth rate assumes there is enough food and space to accommodate the increased population. This cannot be true over long periods of time.

4. The intensity of light at a particular depth is important to divers who wish to estimate visibility and to marine biologists who wish to study the conditions under which plant or animal life thrives. Each depth of a body of water blocks out a certain percentage of the sunlight that hits the top. The percentage that is blocked out depends upon the murkiness and makeup of the water. Suppose a measurement taken on one meter deep shows that light intensity has been reduced by 7%. What is an estimate for the intensity of light 20 meters down?

   Answer: A reduction of 7% corresponds to a size change factor of .93. At twenty meters, the intensity will be multiplied by $.93^{20}$, or approximately .234. Thus only about 23% of the light would get through at that depth.

Comment: The muffling of sound through a medium acts in the same way.

5. Carbon-14 (C_{14}) decays so that in each interval of 5570 years only half of the C_{14} that began the interval is left. What percent of the original would remain in an artifact 10,000 years old?

Answer: The size change factor here is $\frac{1}{2}$, or .5. The unit time period is 5570 years, so 10000 years is about 1.8 unit time periods. So we would expect to have about $(.5)^{1.8}$ or about 29% of the C_{14} to be left.

Comment: C_{14} is employed to date old archeological artifacts or vestiges from the Stone Age. Elements that decay with longer half-lifes are utilized in dating older objects.

Comment: It is common to have the percent of C_{14} given and have to determine the exponent. See note 6.

6. In the area around the University of Chicago, a condominium purchased in 1975 was sold in 1979 for 1 1/2 times its purchase price. What yearly growth rate is that?

Answer: The growth factor for four years was 1 1/2, which we rewrite as 1.5. To find the yearly growth factor, solve

$$x^4 = 1.5.$$

Thus $$x = (1.5)^{1/4} = 1.107.$$

indicating a growth rate of 10.7% per year.

Comment: Growth rates of this magnitude were quite common during the housing boom of the late 1970s. Notice that one does not need to know the buying and selling prices (only the ratio) to calculate the rate.

7. The population of Nigeria grew from 56,400,000 (1964 census) to 66,630,000 (1977 estimate). What yearly growth rate is this?

Answer: Let x be the yearly rate. The growth is in 13 years, so:

$$x^{13} = \frac{66,630,000}{56,400,000}.$$

$$x = \left(\frac{66,630,000}{56,400,000}\right)^{\frac{1}{13}} \approx 1.013.$$

The growth rate was about 1.3% per year.

Comment: Example 7 involves ratio division as well as growth powering. Example 9, Section A, Chapter 10, is similar.

8. What is the effect on prices of 6% yearly inflation over a ten-year period?

Answer: 6% inflation implies a 1.06 growth factor.

$$(1.06)^{10} = 1.7908...$$

1.7908... corresponds to a growth factor of about 79%.

Prices would increase by 79%.

Comment: This amount of inflation was about the case for the U.S. during the 1970s.

9. Inflation rates in the U.S. are often reported by month. If a monthly rate is reported as a 0.7% increase, what yearly rate is equivalent?

Answer: Let x be the yearly (12-month) growth factor. Then, since the monthly growth factor is 1.007,

$$x = 1.007^{12} = 1.0873...$$

for a yearly growth rate of about 8.7%.

Comment: Notice that the yearly growth rate is not 12 times the monthly rate. However, multiplying by 12 gives a good approximation, within reasonable time limits, as the next comment shows.

Comment: Since the reported monthly rate is rounded to the nearest tenth of a percent, the yearly growth factor is more accurately between

$$1.0065^{12} \quad \text{and} \quad 1.0075^{12},$$

i.e., between 1.0808... and 1.0938..., corresponding to yearly inflation rates of between 8.1% and 9.4%. Thus the accuracy for one month is not enough to give much accuracy for the year. Furthermore, inflation rates tend to change from month to month.

10. If a copy machine has a feature which enables it to make copies 2/3 the size of the original (in linear dimensions), what happens if the copies are put through this shrinking 4 times?

Answer: The change is of the factor $(\frac{2}{3})^4$ or $\frac{16}{81}$ or about 20% of the original.

Comment: This is a discrete instance of growth, i.e., one in which there is no interpretation for non-whole number exponents. So it is possible to treat this problem as an instance of size change multiplication, with the size change applied four times, and consider powering only as a notational convenience (see Section C of this chapter).

Powering Use Class C: Notation

The operation of powering has the property that quite large answers can appear even when the base and exponent are routine. For instance,

$$11^{12} = 3{,}138{,}428{,}376{,}721.$$

Similarly, quite small numbers can appear as powers.

$$15^{-8} = .00000000039018442...$$

For these reasons, when very large or very small numbers need to be considered, they are often represented not as decimals in base 10, but as powers.

Powers play a role in several notations that are in common use. The first is the simplest; instead of calculating a number such as 11^{12} above, the number is left as is. This is done when one wants to keep the base and exponent in mind. For instance, on a 10-question True-False test, the probability that two students would randomly choose the same answers is 2^{-10}, showing the connection between the number of questions, the number of choices on each question, and the probability. Were 2^{-10} to be rewritten as the decimal .0009765625, this connection would be lost.

Another common use of powering is in scientific notation and its variants. In these notations, a number is represented as a product of a number between 1 and 10 and an integral power of 10. If there are many significant digits, some of the digits may be ignored. So, for example, the number 11^{12} given above is represented on an SR-50 calculator

as 3.138428377 12

on an Apple as 3.138428378E+12

and in standard scientific notation with three significant digits as

$$3.14 \times 10^{12}.$$

The metric system takes advantage of this property of powering by assigning prefixes to every third power of 10 that occurs reasonably in applications.

Prefix	Power
tera	10^{12}
giga	10^{9}
mega	10^{6}
kilo	10^{3}
----	$10^{0} = 1$
milli	10^{-3}
micro	10^{-6}
nano	10^{-9}
pico	10^{-12}
femto	10^{-15}
atto	10^{-18}

For instance, the nanosecond, a unit often used to measure how long it takes a computer to perform certain operations, is 10^{-9} second. The gigaton, used in estimating the power of some nuclear explosions, is equal to 10^{9} tons.

Several other prefixes are in common use. The prefix deci, as in decibel, stands for 10^{-1}. The prefix centi, as in centimeter, stands for 10^{-2}. The prefix hecto, as in hectare (a unit of land measure), stands for 10^{2}. By having these prefixes, large and small quantities as well as large and small numbers can be expressed easily.

We have already observed in the first use class of this chapter that exponent notation is used in writing various measure units: cm^2 for area, m/s^2 or ms^{-2} (meters per second per second) for acceleration; m^3 for volume, and so on.

Examples:

1. Represent each number in each of the three variants of scientific notation presented in this section.

   (a) 299,809 km/sec, the approximate speed of light.

   (b) 2,878,000,000,000 miles, the approximate distance light travels in a single year (the distance known as a light-year).

   (c) 4,500,000,000, the approximate population of the Earth.

   (d) .0009 m, an approximate width for a human hair.

   (e) a millionth of an inch, the accuracy required in some laser applications.

   Answers: (a) 2.99809 05; 2.99809E+05; 2.99809 x 10^5.

   (b) 5.878 15; 5.878E+15: 5.878 x 10^{15}.

   (c) 4.5 09; 4.5E+09; 4.5 x 10^9.

   (d) 9 -04; 9.0E-04; 9.0 x 10^{-4}.

   (e) 10 -06; 10E-06, 10^{-6}.

   Comment: For (a) and (d) it is reasonable that scientific notation would not be used, since the notation is longer than its decimal equivalent.

2. The following three tables appear in Physics, the PSSC Physics course (Boston: D.C. Heath, 1960). One table covers 10^{-5}m to 10^{18}m; a second covers 10^{19}m to 10^{25}m; the third covers 10^{-6}m to 10^{-15}m. Why were these lengths and distances separated into three sections?

Length in Meters	Associated Distance	Length in Meters	Associated Distance
10^{18}	Greatest distance measurable by parallax	10^{7}	Air distance from Los Angeles to New York
10^{17}	Distance to nearest star		
10^{16}		10^{6}	Radius of the moon
10^{15}		10^{5}	Length of Lake Erie
10^{14}		10^{4}	Average width of Grand Canyon
10^{13}	Distance of Neptune from the sun	10^{3}	One mile
10^{12}	Distance of Saturn from the sun	10^{2}	Length of football field
10^{11}	Distance of Earth from the sun	10^{1}	Height of shade tree
10^{10}	Distance of Mercury from the sun	10^{0}	One yard
10^{9}	Mean length of Earth's shadow	10^{-1}	Width of your hand
	Radius of the sun	10^{-2}	Diameter of a pencil
10^{8}	Mean distance from Earth to the moon	10^{-3}	Thickness of windowpane
	Diameter of Jupiter (Fig. 3–6)	10^{-4}	Thickness of a piece of paper
10^{7}	Radius of Earth	10^{-5}	Diameter of red blood corpuscle

Length in Meters	Associated Distance
10^{25}	Distance to farthest photographed object (a galaxy)
10^{24}	Domain of the galaxies
10^{23}	Domain of the galaxies
10^{22}	Distance to the Great Nebula in Andromeda (nearest galaxy)
10^{21}	Distance to the smaller Magellanic Cloud
10^{20}	Distance of the sun from the center of our galaxy
	Distance to globular star cluster in Hercules (Fig. 3–7)
10^{19}	Distance to the North Star (Polaris)

Length in Meters	Associated Distance
10^{-6}	Average distance between successive collisions (mean free path) of molecules in the air of a room
10^{-7}	Thickness of thinnest soap bubble still showing colors
10^{-8}	Average distance between molecules of air in a room
10^{-9}	Size of molecule of oil
10^{-10}	Average distance between atoms of a crystalline solid
10^{-11}	
10^{-12}	Average distance between atoms packed in center of densest stars
10^{-13}	
10^{-14}	Size of largest atomic nucleus
10^{-15}	Diameter of proton

Answer: The authors wished to make two points. First, that distances in the universe have an extraordinarily wide range. Second, that there are distances too long and distances too short to be measured by rulers, geometry, or light. The second reason accounts for the split into three tables.

BEST COPY AVAILABLE

Comment: The distance to the farthest photographed object has now been estimated as 10^{26}m. Newer microscopes make it possible to more directly measure distances down to 10^{-10}m.

3. If one quantity is approximately 10 times a second quantity, scientists say that the quantities differ by one order of magnitude. So the tables of distances in Example 2 are tables of orders of magnitude. On the next page is a table of orders of magnitudes of times (also from Physics, the PSSC Physics Course, Boston: D.C. Heath, 1960). By how many orders of magnitude do each of these differ?

(a) one minute and one month

(b) the time to write a letter and the time to write a book

(c) the time for light to cross a room and the time for you to cross a room

(d) the time for an electric fan to complete one revolution and the time for a proton to complete one revolution.

Answers: (a) 4 (between 10^2 and 10^6)

(b) 4 (between $10^{7.5}$ and $10^{3.5}$?)

(c) 9 (between 10^1 and 10^{-8} seconds)

(d) 20 (between 10^{-2} and 10^{-22})

Comment: The range of this table is 40 orders of magnitude, the same range as for the table in Question 2.

4. An angstrom, the unit used to measure wave lengths of light, is defined as 10^{-10} meter. (a) How many angstroms are in a meter? (b) In words, what part of a meter is an angstrom?

Answers: (a) 10^{10}, or 10 billion. (b) one ten-billionth.

Comment: By defining the unit to be this small, the numbers with the unit become more manageable. For example, the color blue has wave lengths between 4500 and 5000 angstroms. This is between .00000045 and .00000050 meters.

BEST COPY AVAILABLE

Orders of Magnitude of Times

Each interval is one-tenth of the preceding interval.

Time Interval in Seconds	Associated Event	Time Interval in Seconds	Associated Event
10^{18}	Expected total life of the sun as a normal star	10^{-2}	Time for electric fan to complete one turn
10^{17}	Age of the oldest rocks	10^{-3}	Time for fly to beat its wings once
	Time elapsed since first fossil life		Time that a fired bullet is in the barrel of a rifle
	Time elapsed since first land life		
10^{16}	Time for the sun to revolve around the galaxy	10^{-4}	Time for one vibration of the highest-pitched audible sound
	Age of the Appalachian Mountains	10^{-5}	Time during which firecracker is exploding
10^{15}	Time elapsed since dinosaurs		
10^{14}	Remaining life of Niagara Falls	10^{-6}	Time for high-speed bullet to cross a letter of type
10^{13}	Time elapsed since earliest men		
		10^{-7}	Time for electron beam to go from source to screen in TV tube
10^{12}			
10^{11}	Time elapsed since earliest agriculture	10^{-8}	Time for light to cross a room
	Time elapsed since earliest writing	10^{-9}	Time during which an atom emits visible light
	Time elapsed since the beginning of the Christian Era		
		10^{-10}	
10^{10}	Time elapsed since the discovery of America	10^{-11}	Time for light to penetrate windowpane
10^{9}	Human life span	10^{-12}	Time for air molecule to spin once
10^{8}	Time elapsed since you began school	10^{-13}	
10^{7}	Time for the earth to revolve around the sun (year)	10^{-14}	
		10^{-15}	Time for electron to revolve around proton in hydrogen atom
10^{6}	One month		
10^{5}	Time for the earth to rotate once on its axis (day)	10^{-16}	
		10^{-17}	
10^{4}	Duration of average baseball game	10^{-18}	
10^{3}	Time for light from the sun to reach the earth	10^{-19}	
		10^{-20}	Time for innermost electron to revolve around nucleus in heaviest atom
10^{2}	One minute		
10^{1}			
10^{0}	Time between heartbeats (1 second)	10^{-21}	
10^{-1}	Time for bullet (.30 caliber) to cover the length of a football field (300 ft)	10^{-22}	Time for proton to revolve once in nucleus

5. Two students have exactly the same papers on a 12-question Always-Sometimes-Never test. If they answered the questions randomly, what would be the probability of this occurring?

Answer: 1 in 3^{12}, or 3^{-12}.

Comment: Each possible test paper is a list of twelve A's, S's, or N's. We think of this as 12 slots, each of which can be filled by A, S, or N, that is, in one of three ways. Each of the possible test papers, e.g., AASNASSNNNAS, is a permutation of the 3 objects into the slots. In general, there are m^n permutations of m objects into n slots.

6. A state allows six letters or numbers to be put on its license plates, but does not allow the letter O because it would be confused with the number 0. How many different vehicles can be handled by this system?

Answer: Since less than 6 letters or numbers could be used, think of leaving out a symbol as using the symbol Δ. There are 25 letters (no O), 10 digits, and Δ, or 36 symbols in all. There are 6 slots to be filled, so the number of different license plates is 36^6.

Comment: This number is greater than 2 billion, so there is much room for avoiding obscene or other undesired language.

7. To win a particular lottery, a person must match five digits in a row. What is the probability of doing this?

Answer: We assume that the winning digit combination is picked randomly. There are 5 slots, each of which can be filled by any of 10 digits. So there are 10^5 possible digit combinations, only one of which wins. The chances are 10^{-5} or $\frac{1}{10000}$ or .0001.

Comment: Those who run the lottery hope many people think the chances are better than this.

Summary

The chart summarizes the three use classes of powering discussed in this chapter.

Use class	x	y	x^y
change of dimension	ratio of lengths	dimension	ratio of measures in dimension y
growth	change factor in unit interval	length of interval	change factor in interval of length y
notation	factor	number of times factor is used	power

Except for notation, uses of powering are seldom mentioned in elementary school textbooks.

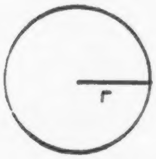
Pedagogical Remarks

Basic facts. As with the other operations of arithmetic, it is helpful for students to memorize some basic facts of powering. These basic facts can be derived (for the student who does not remember them) by using repeated multiplication. In secondary school, it is helpful to know the squares of whole numbers from 1 to 15, 20, 25, 30, 40, . . ., 100. In elementary algebra the student learns that $x^0 = 1$ for any non-zero x and should memorize cubes of whole numbers from 1 to 5 ($1^3 = 1$, $2^3 = 8$, $3^3 = 27$, $4^3 = 64$, $5^3 = 125$.) For second-year algebra and statistics, it is helpful to know powers of 2 with whole number exponents from 4 to 10. ($2^4 = 16$, $2^5 = 32$, $2^6 = 64$, $2^7 = 128$, $2^8 = 256$, $2^9 = 512$, $2^{10} = 1024$.)

Powers of 10 are important to know almost as soon as powering is discussed because of their connections with the decimal system and with scientific notation.

There should be two goals of these first explorations into powering: that the powering notation simplifies many mathematical expressions and that only a few powers are relatively easy to calculate with paper and pencil. Notice how easy it is to calculate powers of 10; this is a good time to associate these powers with the metric system prefixes kilo- (10^3), mega- (10^6), and giga- (10^9); mega is used to measure the power of bombs (megatons), giga is used to measure electrical power (gigavolts). You may wish to connect the powers with the reciprocals of these powers, namely milli- ($1/10^3$ or 10^{-3}), micro- ($1/10^6$ or 10^{-6}), and even nano- ($1/10^9$ or 10^{-9}). The first two are commonly found in measuring amounts of vitamins (milligrams or micrograms), while the last is used in measuring speeds of computers (nanoseconds).

Change of dimension. Begin with the calculation of the area of a square. Ths leads to the formula $A = s^2$ basic to all area calculation. Textbooks generally do a good job of this. Other formulas that involve the second power, such as the formula $A = \pi r^2$ for the area of a circle, can be explained in terms of areas of squares.



Area $= \pi r^2 = 3.14r^2$

Area $= r^2$

About 3.14 of these squares can be fit into the circle at left.

Use lengths of sides large enough so that the area is numerically quite different than the perimeter, and keep units (sq cm for area, cm for length) so that the unit for area is seen to be different than the unit for perimeter.

Volume can be treated next. Find or make open cubes (boxes without the top) with all sides of length 1 unit, 2 units, 3 units, 4 units, etc. Ask how many unit cubes will fit in each? (Answers: 1, 8, 27, 64, which can be written as 1^3, 2^3, 3^3, 4^3.) Fill the cubes with sand and weigh them. The larger cubes should weigh 8, 27, and 64 times the smaller. This shows that weight is related to volume.

Find similarly-shaped objects and compare their lengths, surface areas, and volumes. For example, with similar dolls (i.e., dolls identical except for size), compare their heights, the areas of clothing, and their weights. (One must be careful with the weights, because most dolls are hollow and so weights will vary only like area, not volume.) The results should verify an important theorem from geometry:

> If two figures are similar and a ratio of corresponding lengths in the figures is k, then:
>
> the ratio of any corresponding lengths is k,
> the ratio of any corresponding areas is k^2,
> and the ratio of any corresponding volumes is k^3.

Having done area and volume, one can turn to square root and cube root. The "root" is the side upon which a square or cube stands. That is, a square with area 5 stands on a side whose length is 5, or $5^{1/2}$ or $5^{.5}$ (the decimal tends to disguise the origin of the exponent). A cube with volume 73 stands on a side whose length is 73, or $73^{1/3}$. These two notations, the *radical* notation $\sqrt{\ }$ and the *exponential* notation $(\ )^{1/n}$ are *both* standard, and it is important to learn to move from one to another. Elementary textbooks tend to emphasize radical notation, but exponential notation has the advantage of displaying the properties roots have in common with other powers.

Growth. An important example of growth is compound interest, a good place to begin. As with most applications of powering, work with compound interest used to be inaccessible to elementary school students because the mathematics required to understand the computation was taught only in second-year algebra. Calculators have made this topic not only accessible, but quite easy.

First we must warn you that books do not usually discuss compound interest in a way that is helpful for computation. A typical book will solve an interest problem by using addition. If $200 is invested at 6% for a single year, the book will suggest that one calculate the interest and add it to

$$.06 \times \$200$$

$200, for a total of .06 x $200 + $200. This is fine for the first year, but then what do you do the second year? We feel that the traditional method is much too complicated. It is far easier never to calculate the interest, but concentrate on what total there will be at the end of the first year, namely

$$1.06 \times \$200$$

and thus the size change or growth factor 1.06 is staring the reader in the face. After the second year, at this rate, there will be

$$1.06^2 \times \$200$$

and the pattern is set. The third year, one will multiply again by 1.06.

$$1.06^3 \times \$200$$

and now it is easy to generalize. In n years, there will be

$$1.06^n \times \$200.$$

The wonderful thing about this formula is that n, being a measure, does not have to be a whole number. If the interest is compounded continually (or daily, as banks often advertise), and you want the interest after one month (1/12 of a year), just calculate

$$1.06^{1/12} \times \$200.$$

If you are worried about putting a fraction into the calculator, enter the almost-equivalent decimal .0833333.

After doing some calculations of compound interest, it is important to take some realistic but high rate (for example, 18%, a rate often used for overdue charge accounts), and consider what an amount will grow to at this rate. Here is what happens to $1000 at this rate.

after 1 year: $\$1000 \times 1.18 = 1180$

after 2 years: $\$1000 \times 1.18^2 = 1392.40$

after 3 years: $\$1000 \times 1.18^3 = 1643.03$

after 4 years: $\$1000 \times 1.18^4 = 1938.77$

after 5 years: $\$1000 \times 1.18^5 = 2287.75$

It only takes a little over 4 years to double at this rate! Then consider a different rate such as the current inflation rate, and ask what will happen to the cost of a car, or a home, in 5 years at this rate.

Questions

1. (a) Give the volume of a box that is 15 cm on a side. (b) How does that volume compare with the volume of a second box that is 30 cm on a side?
2. A $2\frac{1}{2}$ year certificate of deposit advertises an annual rate of 12%. How much will $10000 grow to in that time? (The answer is not $13000.)
3. Each inch of insulation keeps in 60% of the heat behind it. What percentage will be kept in by 8 inches of insulation?
4. What is the probability of correctly guessing all 10 answers on an always-sometimes-never test?
5. A car purchased for $3000 in 1968 is roughly equivalent to one purchased for $6000 in 1980. What is the yearly rate of change in the price?
6. At 10% inflation, what happens to the cost of an item in 7 years?
7. The membership in an organization grew from 400 to 500 in a 3-year period. What is the yearly growth rate?
8. A 12-oz drinking glass sold at a store has a circular base with diameter 2 3/4". The store clerk indicates that a similarly-shaped 16-oz glass is available. What is a likely diameter of the base of this larger glass?
9. How much more should a person 7-feet tall weigh than one 6-feet tall?
10. Write as a decimal. Can either quantity be expressed with numbers between 1 and 1000 by using a different unit?
    (a) 3.9×10^{26} watts, an estimated continuous output of power of the sun.
    (b) 10^{-7} watts, the power produced by the sound of an ordinary conversation.

Notes and Commentary

1. Powering as an operation
2. Powering as a notation
3. Powering as repeated multiplication
4. Powering as more than repeated multiplication
5. Work of others
6. Related facts
7. Recovering exponent as a use class
8. Other use classes

1. *Powering as an operation*. We have given the name *powering* to the binary operation that maps (a,b) onto a^b. Another name given to this operation is *exponentiation*, but quite often no name is given. Computer language requirements have forced creations of symbols to represent this operation rather than the juxtaposed right superscript idea a^b. At least three symbols are in common use:

a ** b	a ↑ b	a ^ b
FORTRAN	BASIC	APPLESOFT BASIC

Computing powers is not equivalent to discussing the operation of powering. Books tend to avoid mentioning notions such as the lack of commutativity (a^b ≠ b^a) or associativity (a^b)^c ≠ a^(b^c), or the existence of a right identity (for all x, x^1 = x) but not a left identity. Though examples are customarily given showing the lack of distributivity of powering over addition (a^(b+c) ≠ a^b + a^c) and (a+b)^c ≠ a^c + b^c), these are not usually tied to any analysis of the operation itself.

2. *Powering as a notation*. If powering is not treated as an operation, then how is it treated? We view the standard developments as treating powering as a notation. Explicitly, this is what we mean.

In the elementary school, the notation a^b is introduced as a shorthand for the repeated multiplication using *a* as a factor *b* times. So at first, the symbol has meaning only when *b* is a positive integer. Square roots, though they could be considered as powers, are not treated as such because the repeated multiplication conception does not allow *b* to be 1/2. Thus the student encounters x^1/2 in the form of $\sqrt{x}$ but is not told about the connection between them.

When it is finally desired to allow *b* to take on zero as a value, or negative integers, or any rational number, then the symbol is defined at each juncture. Thus b^0 is *defined* to be 1, *not deduced* to be 1 from the general property x^m . x^n = (x^m+n). Some books do show that we must define b^0 = 1 if we wish the general property to hold, but there is not the same treatment that is given multiplication, where 0a is deduced to be equal to 0 from the distributive property ma + na = (m+n)a.

BEST COPY AVAILABLE

Similarly, b^-n is defined to be the reciprocal of b^n, even though one could prove that these numbers are reciprocals if one started from the general property given above. And similarly, b^(m/n) is defined in terms of roots and powers previously considered. When it comes to real numbers as exponents, as in b^ , these too are defined.

The alternate procedure is to take powering as a basic operation like multiplication and addition, and assume the properties of powering just as properties of multiplication and addition are assumed. The properties of powering do correspond to the properties of these other operations (Usiskin, 1974), and such a development has been done with little noticeable difference in student performance but some saving of teaching time (Usiskin, 1973, 1975). The basic properties are three: For all x > 0 and y > 0, and all real m and n,

x^m . x^n = x^m+n
(x^m)^n = x^(mn)
(xy)^m = x^m . y^m.

Even this development has to separate negative bases as a special case, for properties that hold when x > 0 do not necessarily hold when x < 0.

3. *Powering as repeated multiplication*. As with all operations in school arithmetic, the understanding of powering has been strongly linked to the ability to work with paper and pencil to calculate powers. When *b* is a small positive integer, a^b can be calculated by resorting to repeated multiplication, so powering has been introduced as just that: repeated multiplication. The student is taught to view powering as a shorthand: x^2 is short for x.x; x^3 is short for x.x.x, and so on.

While this conception of powering is ah important mathematical conception to have, reliance upon it as the *sole* conception of powering has led to difficulties. The student looks upon even simple expressions as 3^-2 as mathematical contrivances or tricks and has little if any intuition for them. There is, for instance, no notion that since 3/5 and 5/8 are relatively close to each other, one should expect x^(3/5) to be rather close to x^(5/8). A student is as likely to believe that x^0 = 0 as to believe that x^0 = 1.

Teachers are aware of these difficulties and tend to ascribe them to the difficulty of the operation, not to the incorrectness of the pedagogy.

We have earlier mentioned that treating multiplication solely as repeated addition provides a narrow view of multiplication. Yet at least we acknowledge that multiplication is an operation with its own properties. Powering deserves the same.

BEST COPY AVAILABLE

In this the calculator plays a significant role. Powering wit non-integer exponents has traditionally been introduced in second-year algebra, because only at that time had the student acquired the means (i.e., logarithms) to calculate with them. Now even inexpensive calculators have an $x^{\wedge}y$ or equivalent key and the calculation is not a problem. The teacher who feels that using calculators is inappropriate and will lead to less understanding is forgetting two things. First, log tables themselves are artificial aids for computation; the student is using numbers from these tables on faith, for the student has not participated in their calculation and probably does not know how they were derived. Second, the understanding of logarithms has always been poor at best among students. The quicker calculation that calculators give may enable the teacher to have more time to deal with issues of understanding.

Finally, we repeat that treating powering as repeated multiplication makes it impossible to encompass the uses of powering.

4. *Powering as more than repeated multiplication*. In the discussion beginning the chapter, we pointed out that any expression of the form

$$x^{\wedge}y = z$$

is equivalent to

$$x = z^{\wedge}(1/y)$$

and expressions of the form $a(x^{\wedge}y) = z$ and $a = z(x^{\wedge}{-y})$ are also equivalent, so that if y is a whole number, one is forced into consideration of unit fractional and negative exponents. It's common to think of this as being *the* sequence for powering applications, that one always begins with whole number values for y.

This is not the case. All uses of powering have instances where the more common values of y are not whole numbers. What complicates the issue is that there are always uses where y is a whole number.

For example, regarding changes of dimension, those uses that begin with area and volume have y as a whole number 2 or 3, but fractals (see note 6) seldom operate so simply. If interest is compounded daily, quarterly, or in some other periodic fashion, y can be treated as a whole number, but if interest is compounded continuously, y is virtually never a whole number. In calculating permutations, one may use factorials (see Chapter 7, Section B, Example 8), but in statistics the gamma function generalizes this notion to cases where the arguments are not integers.

5. *Work of others*. We know of no attempt by others to classify uses of powering. However, examples of powering uses do appear at many levels of mathematics study. Area and volume formulas involving exponents 2 and 3 appear as early as middle

school. These are discussed in more detail in geometry courses (see the pedagogical comments dealing with change of dimension). Growth applications are common in second-year algebra texts (see partcularly Foerster, 1980). Permutation uses are found in many 11th and 12th grade mathematics books and in all elementary probability texts. A particularly good source of problems utilizing powering is to be found in A Sourcebook of Applications of School Mathematics (Bushaw et al., 1979).

6. Related facts. Because powering is not commutative, each powering fact determines two related facts that involve quite different operations. Whereas all related facts of addition and multiplication can be expressed in terms of addition and multiplication themselves, as seen in the chart below, one of the related facts for powering involves logarithms, and so only for powering do there exist situations in which the operation itself cannot be used to recover one of its own components.

Given	Related Facts	Expressed in terms of given operation
addition	$x = z - y$	$x = z + -y$
$x + y = z$	$y = z - x$	$y = z + -x$
multiplication	$x = z \div y$	$x = z \cdot 1/y$
$xy = z$	$y = z \div x$	$y = z \cdot 1/x$
powering	$x = z$	$x = z$^(1/y)
x^y = z	$y = \log_x z$	cannot be done

7. Recovering exponent as a use class. The bottom line in the chart shows that if one has a powering situation in which the exponent is not known, logarithms will probably be needed for the answer. A use class involving the recovery of an exponent could have been placed in the powering chapter, but we decided to place our remarks here due to the broader mathematical experience which such a discussion entails.

Generally, if a^b = c, then $b = \log_a c = \frac{\log c}{\log a}$.

Thus one uses logarithms (easily found on a scientific calculator) to answer questions like the following.

1. (from Change of Dimension) In the change of dimension use class, one usually knows the dimension, so it is not common to have to solve for the exponent. But what is the dimension of a coastline in which the following happens?

   A rough coastline is to be measured in yards. Dividers a yard wide are set up and one walks along and counts how many times the dividers have to be moved. Suppose the length of a part of coastline is thus found to be 1000 yards. Now suppose a measurement is made in the same way with dividers a foot apart. Since there are three feet in a yard, one would expect the length of coastline to be 3000 feet. But in a

BEST COPY AVAILABLE

rough coastline this does not happen. The foot dividers go into bends and crevices that the yard dividers miss, and there will be more than 3000 feet of "length". In a rough coastline, it can can happen that 4000 feet will be counted where only 1000 yards were. (This is not far off experimental values found for Great Britain.) What's the dimension in such a case?

Answer: Examine the pattern.

To change length from yards to feet, multiply by 3^1.
area 3^2.
volume 3^3.
coastline from yards to feet, multiply by 4.

The exponent of 3 is always the dimension. so we write 4 as 3^d. $4 = 3^d$. Then $d = \frac{\log 4}{\log 3} \approx 1.26$, something between length and area. We conclude that the coastline being measured is of dimension 1.26.

2. (From Change of Dimension) Give the dimension of a range on the side of a hill if, when area is measured in ares (1 are = 1 square dekameter = 100 square meters), the region has area 450 ares; but if the area is measured in m^2, its area is 50000 m^2.

Answer: To change ares to m^2, we expect to multiply by 100. That is, to change dekameters to meters in dimension d, we multiply by 10^d.

$$\text{Here} \quad 10^d = \frac{50000}{450}$$

$$d = \log \frac{50000}{450} \approx 2.046$$

The dimension is only a little over 2. The region is in rolling country.

Examples 1 and 2 utilize ideas from the landmark book Fractals (Mandelbrot, 1976), in which a variety of phenomena (coastlines, Swiss cheese, radio interference, turbulence, etc.) are shown to have fractional dimension.

3. (from Growth) At an annual inflation rate of 12%, how long would it take for the price of an item to double?

Answer: The size change factor is 1.12, the unit interval one year. Doubling connotes a size change factor of 2. If y is the number of years, we wish to know when

$$1.12^y = 2$$

BEST COPY AVAILABLE

This implies $y = \frac{\log 2}{\log 1.12}$ 6.1

At this rate prices will double in a little over 6 years.

4. (from Growth) If 18% of C14 is present in an artifact, how old is the artifact?

Answer: The size change factor is 1/2, the "half-life" interval for C14 is 5570 years. If a is the number of such intervals, we wish to know when

$$(1/2)^a = 18\%$$

$$a = \frac{\log .18}{\log .5} \quad 2.47$$

The age of the artifact is

$$2.47 \text{ intervals} \times \frac{5570 \text{ years}}{\text{interval}} \text{ or}$$

about 13,800 years.

5. (from Notation) A public health researcher wishes to code each of the 100,000 people in a study by a sequence of letters of the English alphabet. To save computer space, as short a sequence as possible is to be used. How short a sequence will suffice?

Answer: Let n be the number of letters in the sequence. We need

$$26^n > 100{,}000$$

$$n > \frac{\log 100{,}000}{\log 26} > 3.53$$

All of the people can be coded with the use of only 4 letters. (Since $26^4 = 456{,}976$, many more people could be coded with the same number of letters.)

6. (from Notation) For a multiple-choice test with 5 choices for each item, how many items are needed to insure less than a 1 in 1000 chance of getting all items correct by guessing?

Answer: Let n be the number of items. There are 5^n permutations of responses, only one of which is correct. So there is a $1/5^n$ or 5^{-n} chance of getting all items correct. We wish to know when 5^{-n} is less than 1/1000.

BEST COPY AVAILABLE

$$5^{-n} < 1/1000$$

$$\text{exactly when } 5^{n} > 1000$$

$$n > \frac{\log 1000}{\log 5} \quad 4.3$$

So 5 items are needed.

8. Other use classes. We avoided an entire chapter (not just a section) by placing instances of recovering base with their corresponding uses. That is, we characterized uses that lead to a situation of the form $x = z^{(1/y)}$ with their equivalent $x^y = z$. Thus, for example, we placed finding the side of a cube given its volume in the same section as finding the volume of a cube given its side. This decision was based upon a belief that separating roots from powers with respect to uses is equivalent to separating out uses of positives from uses of negatives in addition, or uses of whole numbers from uses of fractions in multiplication. While pedagogically the separation is a wise thing to do in first introductions, the goal should be general conceptions that blur distinctions between the specific numbers involved.

We considered a use class that would cover instances of permutation and likelihood. This consideration was motivated by the existence of two types of expressions in which all sorts of powers may be involved. One type involves the gamma function Γ, a function with the property that

$$\Gamma(n+1) = (n+1)\,\Gamma(n)$$

for all real numbers n, and so a function that generalizes the factorial function. The gamma function has many applications in probability theory and related areas. A second type involves the many probability functions that utilize powering. For instance, the Poisson distribution is defined by

$$f(x) = \frac{m^{x} e^{-m}}{x!}$$

where x is a non-negative integer but m is a parameter that is seldom an integer. The normal distribution involves an expression of the form

$$e^{\frac{-x^{2}}{2}}$$

where x is seldom an integer. Generally, we considered such examples too advanced for this book and avoided the use class entirely.

BEST COPY AVAILABLE

We are aware that there are also fractional derivatives and integrals and that these may have important applications, though we admittedly do not understand them. These applications could be construed as generalizing the acting across uses of multiplication and rate uses of division.

BEST COPY AVAILABLE

CHAPTER 10

USES THAT COMBINE OPERATIONS

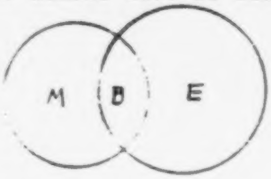
Many applications involve more than one of the fundamental operations, or involve the same operation in more than one way. Hence, the focus of Chapters 5 through 9 on single operation use classes cannot suffice for a complete understanding of the uses of operations in applications. One approach to situations involving more than one use class is to start with the collection of use classes and seek ways of combining them. Another approach is to examine applications that involve more than one operation and ask if they can be broken down into individual steps that themselves embody the use classes of earlier chapters. In these ways, use classes are analogous to basic properties of mathematical systems. Just as axioms in mathematical systems can be put together to deduce theorems, simple uses can be put together to form more complicated uses. And just as complex properties can be traced to first principles, complicated applications can be broken down into simple uses.

There are obviously many more possible combinations of use classes than there are use classes themselves. It would greatly increase the length of this book if we tried to explore all of the various combinations of the use classes we have presented earlier. For a complete analysis, we would have to consider also the use classes from Parts I and III as well. From a practical point of view, it is impossible for us to do this.

In order to avoid this complexity, yet show a representative collection of applications involving more than one use class, we have organized this chapter in the following way: Section A gives examples of applications involving exactly two use classes. Section B includes examples of applications involving more than two use classes. Section C presents examples of applications that are not easily separable into constituent use classes. Clearly, in no section are the examples meant to exhaust the possible types of applications, but we have tried to include prominent uses.

Section A: Applications Involving Exactly Two Use Classes

A school has honors programs in English and mathematics and wishes to determine how many students are involved in these programs. One way of calculating is to add E, the number of students in the English honors program to M, the number of students in the mathematics honors program. Those B students in both programs simultaneously are counted twice, so their number must be subtracted.



For instance, if E = 60, M = 48, and B = 11 (meaning that 11 students are in both programs), then the number of students involved is

$$60 + 48 - 11 \text{ or } 97.$$

The addition is putting together; the subtraction take-away. What we have here is an instance of the combining of two use classes in a single situation.

The examples that follow have been chosen to display all of the operations of earlier chapters. The reader would probably gain most by answering each question and trying to identify possible use classes before reading the answers and comments. In the comments, we suggest specific use classes knowing that other interpretations may exist.

Examples:

1. Linear combination. Pencils are being sold for 10¢ each and erasers for 5¢ each. How much will it cost for 6 pencils and 2 erasers?

BEST COPY AVAILABLE

Answer: $6 \text{ pencils} \times 10 \frac{\text{cents}}{\text{pencil}} + 2 \text{ erasers} \times 5 \frac{\text{cents}}{\text{eraser}}$

$= 60 \text{ cents} + 10 \text{ cents}$

$= 70 \text{ cents}$

Comment: The units signal the use meanings. The costs for individual pencils and erasers are rate factors; they are multiplied by the numbers of pencils and erasers bought and the products added to get the total cost. The use meanings are rate factor (multiplication) and putting together (addition).

Comment: There are three operations performed here. We classify the problem in this section because only two use classes are involved.

2. Fixed cost and constant increment. A certain car rental agency charges $35 a day and 30¢ a mile to rent a car. What will it cost to rent and drive this car 80 miles in a single day?

Answer: $\$35 + 80 \text{ miles} \times \frac{\$.30}{\text{mile}} = \$59.00$

Comment: The multiplication use is rate factor; the addition use is putting together.

3. Averages. In a school, in a particular week, the following numbers of copies have been made on a duplicating machine:

Monday	. 245 copies
Tuesday	130 copies
Wednesday	117 copies
Thursday	460 copies
Friday	1015 copies

What is the average number of copies per day?

Answer: Add up the number of copies and divide by 5.

$$\frac{1967 \text{ copies}}{5 \text{ days}} = 393.4 \frac{\text{copies}}{\text{day}}$$

Comment: The addition exemplifies putting together. The division is a rate, the expression "copies per day" giving it away.

Comment: All simple averages can be considered as rates, the result of distributing a total over a number of entries, i.e., a total per entry.

4. Rate of change. At the beginning of a diet, Jorge weighed 60 kg. Two weeks later he weighed 57.5 kg. How fast has he been losing weight?

Answer: To calculate an answer in kg per day, first find the amount lost, then divide by 14 days. The answer is about .18 kg per day, or about 180 grams per day.

The calculation can be written as

$$\frac{\text{change in weight}}{\text{time}} = \frac{\text{present weight} - \text{former weight}}{\text{time}}$$

$$= \frac{(57.5 - 60)\text{kg}}{14 \text{ days}}$$

$$= \frac{-2.5 \text{ kg}}{14 \text{ da}} \approx -.18 \frac{\text{kg}}{\text{da}}$$

Comment: The negative answer shows that a weight loss has taken place. The subtraction is comparison; the division is rate.

5. Slope. The 1954, Roger Bannister of Great Britian first ran a mile in less than 4 minutes, with a time of 3:59.4 (three minutes, 59.4 seconds). In 1982 the world record, held by Sebastian Coe, also of Great Britain, was 3:47.3. What was the average change in this record (in seconds per year) from 1954 to 1982?

Answer: Calculate the change in time and the change in years, then divide to get the rate of change.

$$\frac{\text{change in record}}{\text{change in years}} = \frac{\text{present record} - \text{former record}}{\text{present year} - \text{former year}}$$

$$= \frac{3{:}47.3 - 3{:}59.4}{1982 - 1954}$$

$$= \frac{-12.1 \text{ seconds}}{28 \text{ years}}$$

$$\approx -.43 \frac{\text{seconds}}{\text{year}}$$

The record has been going down at almost a half second a year, on the average.

Comment: This problem is conceptually like the weight loss problem (Example 3), except that two subtractions are needed.

Comment: In algebra and analytic geometry, the result of the calculation done here is called slope. We think one reason some algebra students have trouble interpreting the slope formula

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

as rate of change is that they have never been taught to associate change with subtraction ($y_2 - y_1$ and $x_2 - x_1$) or rate with division.

6. Proportions. A recipe for 5 people calls for 1/2 tsp. salt. To enlarge this recipe for 8 people, how much salt is needed?

Answer: To "scale up" this recipe, divide to get the size change factor $\frac{8}{5}$. Then multiply all ingredients by $\frac{8}{5}$. Needed are

$$\frac{8}{5} \times \frac{1}{2} \text{ tsp} \quad \text{or} \quad \frac{4}{5} \text{ tsp},$$

or a little less than 1 tsp. of salt.

Comment: The two use classes illustrated in this answer are ratio (division) and size change (multiplication), and require no algebra. To use algebra, a proportion could be set up using equal ratios,

$$\frac{5 \text{ people}}{8 \text{ people}} = \frac{1/2 \text{ tsp}}{x \text{ tsp}}$$

or equal rates,

$$\frac{1/2 \text{ tsp}}{5 \text{ people}} = \frac{x \text{ tsp}}{8 \text{ people}}$$

The second (tsp per person) seems easier for more students to understand.

7. Proportions. Five cans of juice cost \$2. At this price, how much will 8 cans cost?

Answer: Calculate unit cost (a rate).

$$\frac{\$2}{5 \text{ cans}} = \$.40 \text{ per can}$$

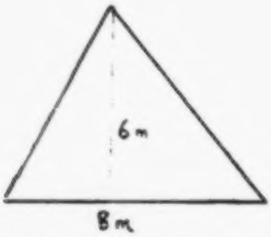
Multiply the number of cans by the cost per can to get total cost.

$$8 \text{ cans} \times \$.40/\text{can} = \$3.20$$

Comment: The two uses classes in this solution are rate (division) and rate factor (multiplication). Except for context, this example is similar to Example 6. The solutions exhibit several ways of answering such problems. In Example 6, we illustrated two solutions with ratios and one with rates. The solution for Example 7 uses unit rates. Which method is more natural depends on the context, the numbers in the situation, and the solver.

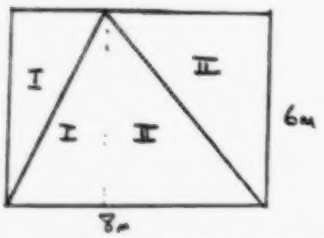
Comment: Sci-Math (Goodstein et al, 1982) contains an in-depth discussion of the solving of proportions using rates and ratios.

8. Area. In yachting races, sail area for a given class of boat is often limited. What is the area of a triangular sail 6 meters high and 8 meters long?



Answer: There is a well-known formula, $A = \frac{1}{2}bh$. From this the area is $\frac{1}{2} \cdot 8\text{m} \cdot 6\text{m}$, or 24m^2.

Comment: If you did not know the formula, you could think of a rectangular sail first. The area of that sail is 48 m^2 (multiplication acting across). Splitting that into 2 parts equal in area, as the drawing above shows, we get (by rate) $\frac{48 \text{ m}^2}{2 \text{ parts}} = 24 \frac{\text{m}^2}{\text{part}}$, so 24 m^2 for the area of the triangle.



Comment: Another way to conceptualize the situation is that the ratio of the areas of the triangle and rectangle is $\frac{1}{2}$ (ratio division), so the area of the triangle is $\frac{1}{2} \cdot 48\ m^2$ (multiplication size change) or $24\ m^2$. This shows that the multiplications done in $A = \frac{1}{2}bh$ can be traced back to simple uses of the operations. This is true of all measurement formulas.

9. In a school district known to the authors, the salary for a first-year teacher with a master's degree was \$5600 in 1963-64 and \$16,000 in 1981-82. What is the yearly growth rate over this time interval?

Answer: Let x be the yearly growth factor.

Then, since the interval is 17 years long:

$$x^{17} = \frac{\$16000}{\$5600} \approx 2.857, \text{ the 17-year growth factor.}$$

Taking the 17th root of each side,

$$x \approx (2.857)^{1/17} \approx 1.064.$$

There was about a 6.4% growth rate per year.

Comment: This is close to the inflation rate over that time, meaning that the teacher's salaries in this district kept pace with inflation.

Comment: This example combines the use of ratio division with the growth use of powering. The combination is common because a growth factor (the base in the powering) compares quantities before and after growth.

10. *A trick problem*. A car travels 100 miles at 25 mph and returns going 50 mph. What is its average rate?

Comment: In this kind of problem, the word "average" is a misleading cue, for the average of 25 mph and 50 mph is not what is desired. We have purposely kept the semantics of the way the problem is customarily stated--otherwise the problem would not be so well-known.

Answer: What is desired is the rate for the entire trip. From the rate meaning of division:

$$\text{rate in mph} = \frac{\text{distance in miles}}{\text{time in hours}}$$

So $$\text{rate of trip} = \frac{\text{total distance}}{\text{total time}}$$

and $$\text{time} = \frac{\text{distance}}{\text{rate}}$$

The last of these is the rate divisor use class. The time going is $\frac{100 \text{ miles}}{25 \frac{\text{miles}}{\text{hour}}}$ or 4 hrs. By the same process, it takes 2 hours to return. So

$$\text{rate of trip} = \frac{100 \text{ miles} + 100 \text{ miles}}{4 \text{ hrs} + 2 \text{ hrs}} = \frac{200}{6} \text{ mph} = 33\tfrac{1}{3} \text{ mph.}$$

(The additions are both putting together measures.)

Comment: The rate is twice as close to 25 mph as it is to 50 mph because the car travelled for twice the time at 25 mph as it did at 50 mph. Thus the answer can be interpreted as the weighted average:

$$\frac{4 \text{ hours} \times 25 \text{ mph} + 2 \text{ hours} \times 50 \text{ mph}}{6 \text{ hours}}$$

See Example 1, Section B for another example of a weighted average.

Section B: Applications Involving More than Two Use Classes

The idea represented in this section is just like that in the last, except that here more than two use classes are involved.

Examples:

1. Weighted average. In many schools, grades are worth points as follows: A = 4 points, B = 3 points, C = 2 points, D = 1 point, and E = no points. If a student receives one A, two B's, one C and one D, what is the student's grade point average?

   Answer: One multiplies the number of grades by the points per grade (rate factor), adds these (putting together measures) and then divides (rate) by the number of courses to get

   $$\frac{1 \times 4 + 2 \times 3 + 1 \times 2 + 1 \times 1}{5} = \frac{13}{5} = 2.6$$

   which is the average grade per course, the grade point average. With units included, we have

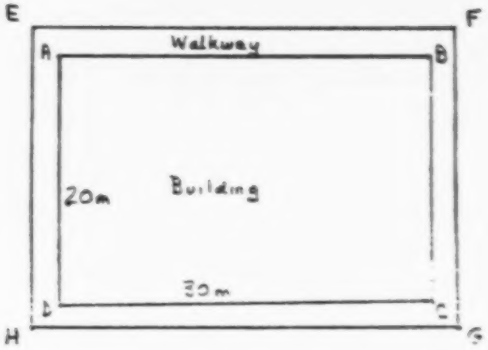
   $$\frac{1 \text{ grade} \times 4 \, \frac{\text{points}}{\text{grade}} + \ldots}{5 \text{ courses}} = \frac{13 \text{ points}}{5 \text{ courses}} = 2.6 \, \frac{\text{pts}}{\text{course}}$$

   Comment: It is also possible to analyze the multiplication as scalar multiplication, where the grades themselves are equal to numbers from 0 to 4, not points. In fact, some schools use numbers, rather than letters, as grades.

   Comment: Weighted averages can be used to calculate the average interest rates on investments at different rates, the average rate of an automobile that has gone different distances at different times (Example 10 in Section A), etc.

BEST COPY AVAILABLE

2. Area. A walkway 2 m wide is planned around a building whose shape is a rectangle 30 m by 20 m. To determine how much concrete is needed for the walkway, a contractor needs to know its area. What is the area?



Answer: One way to determine the area of the walkway is to subtract the floor area of the building from the area of the rectangle that includes the building and the walkway, i.e., to subtract the area of ABCD from the area of EFGH. To get the dimensions of EFGH, add 2m on each side of the building. So EFGH has dimensions 34 m and 24 m. Its area is 34 x 24 m^2, or 816 m^2. ABCD has area 600 m^2. The walkway then has area 816 m^2 - 600 m^2, or 216 m^2.

Comment: Use classes applied here are putting together addition, take-away subtraction, and acting across multiplication.

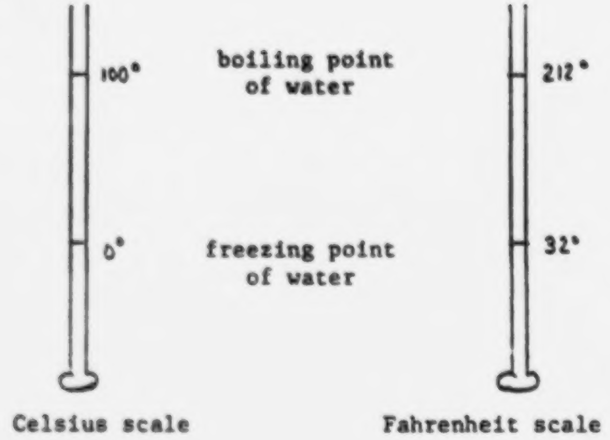
Comment: Another way is to split the walkway into 4 rectangles, 2 on the side with dimensions 24 m x 2 m, and a top and bottom 30 m x 2 m. This avoids subtraction.

Comment: Many people estimate too low for the area of the walkway. It is narrow but quite long.

3. Derive the formula $C = \frac{5}{9}(F - 32)$, which translates Fahrenheit temperatures (F) into Celsius temperatures (C), in terms of use classes.

BEST COPY AVAILABLE

Answer: Examine the drawing showing the corresponding key points on the scales.


100°
boiling point of water
212°
0°
freezing point of water
32°
Celsius scale
Fahrenheit scale

To compare these scales as measures, we must shift the Fahrenheit value down 32° so the zeros will correspond. That subtraction shift explains

$$F - 32.$$

Now, for the intervals between boiling and freezing points of water (subtraction comparison), there are 100 Celsius degrees for every 180 Fahrenheit degrees. Dividing these (ratio or rate division) gives us the necessary factor $\frac{100°C}{180°F}$ or $\frac{5°C}{9°F}$. Thus

$$\text{degrees Celsius} = \frac{5°C}{9°F}(\text{degrees Fahrenheit} - 32)$$

or, as is usually stated

$$C = \frac{5}{9}(F - 32)$$

Comment: The units in the constant $\frac{5°C}{9°F}$ are necessary, as is often the case with constants in formulas, so that the quantities on both sides of the equal sign will have the same units.

BEST COPY AVAILABLE

4. Compound interest. \$10000 is invested at 8.234% annual rate for four years. How much income should be expected at the end of that time?

Answer: An 8.234% annual rate corresponds to a 1.08234 yearly growth factor, since $1 + 8.234\% = 1.08234$. Now raise to the 4th power (growth) to get the growth factor for four years.

$$(1.08234)^4 \approx 1.3723182$$

Apply this to the quantity \$10,000 (size change multiplication).

$$\$10000\ (1.3723182) \approx \$13,723.18$$

This gives the value of the investment. Subtract the original 10,000 (either take-away or comparison) to calculate income.

$$\$13,723.18 - \$10,000 = \$3,723.18$$

So, \$3,723.18 should be expected as income.

Comment: Putting all the steps together,

$$\$3,723.18 = \$10,000\ (1 + .08234)^4 - \$10,000$$

This gives the form of the general formula, where P is the principle (original amount), r the unit interval rate, and t the number of intervals:

$$\text{Income} = P\ (1 + \frac{r}{100})^t - P$$

Thus what seems to be a very complicated formula can be analyzed rather completely by use class ideas. The one part of the formula not analyzable in this way is $1 + \frac{r}{100}$. The $\frac{r}{100}$ comes from the definition of percent and the adding of 1 is a byproduct of the distributive property

$$P + Px = P(1+x).$$

5. Annuities. An individual puts \$2000 annually into a retirement fund. If the fund guarantees 8% interest a year on what is deposited in it, to how much will the individual's investment have grown after 20 years?

Answer: After one year, using the reasoning of Example 4, there is \$2000 x 1.08. After two years, the first deposit has grown $\$2000 \times 1.08^2$ while the second year's deposit is just \$2000 x 1.08. After three years, there is

$$\underbrace{\$2000 \times 1.08^3}_{\text{from first year's \$2000}} + \underbrace{\$2000 \times 1.08^2}_{\text{from second year's \$2000}} + \underbrace{\$2000 \times 1.08}_{\text{from third year's \$2000}}$$

After twenty years, 20 quantities must be added:

$$\$2000 \times 1.08^{20} + \$2000 \times 1.08^{19} + \ldots + \$2000 \times 1.08.$$

With a calculator, we found the total to be:

\$100,845.84

of which all but \$40,000 is interest.

Comment: Similar calculations are used to determine how much should be paid on loans and mortgages. Fortunately there exists a formula for adding up a number of quantities like those found in the answer.

$$ar^n + ar^{n-1} + ar^{n-2} + \ldots + ar = \frac{ar^{n+1} - ar}{r - 1}$$

In Example 6, $a = \$2000$, $r = 1.08$, and $n = 20$.

Comment: The use classes are powering growth, size change multiplication, and putting together addition.

BEST COPY AVAILABLE

Section C: Applications not Readily Separable Into Constituent Use Classes

As an application involves more arithmetic operations, it becomes more difficult to keep track of the operations, and so it becomes more likely that a formula will exist that aids in doing the calculations. The later examples of Section B are complicated enough so that most people use formulas that codify what is to be done.

Every formula that involves many operations can be broken down in some way into a succession of single operations; after all, we compute values in such formulas by doing the operations one at a time. In the previous two sections, these single operations reflected use classes. In this section, we exhibit formulas in which it is not so easy to determine the particular use class of every constitutent operation.

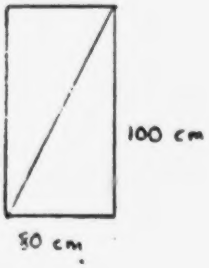
The difficulty in breaking down formulas into use meanings is caused by three factors. The formula may have been derived using mathematical properties rather than use meanings of the operations, thus hiding use meanings that may have been originally involved (see Example 1). The formula may involve use meanings that require mathematics more advanced or more intricate than that with which the user has familiarity, forcing the user to accept the formula blindly rather than being able to assimilate it into a known context or use class (see Example 2). The formula may be an approximation, so there may be reasons why it is a good approxmiation without necessarily a strict compatibility between the operations in the formula and the uses of that formula (see Example 3). Generally, as one becomes more familiar with use classes, and learns more mathematics, one becomes able to explain more formulas. Thus an application "not readily separable into constituent use classes" for one person may be one that a second, more knowledgeable person, can separate, just as in mathematical systems, some people are able to prove theorems better than others.

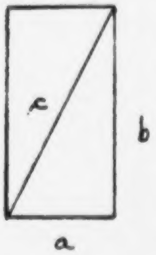
Examples:

1. Pythagorean Theorem. A diagonal brace is to be placed on a door that is 22 cm high and 80 cm wide. What will be the approximate length of the brace?

   Answer: The famous Pythagorean theorem helps to answer this question. In any right triangle, $c^2 = a^2 + b^2$, or equivalently

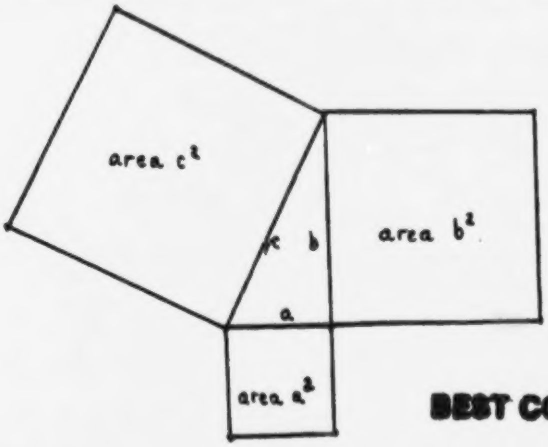
   $$c = \sqrt{a^2 + b^2}.$$





Applied to this situation,

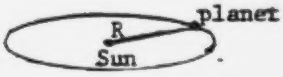
$$c = \sqrt{200^2 \text{ cm} + 80^2 \text{ cm}} = \sqrt{46400^2 \text{ cm}} \approx 215 \text{ cm}.$$



BEST COPY AVAILABLE

Comment: For the ancient Greeks, the Pythagorean Theorem was seen as a relationship between the areas of the three squares drawn on the sides of a right triangle (see above), namely $a^2 + b^2 = c^2$. Thus, for the Greeks, the formula utilized the acting across meaning of multiplication and the putting together meaning of addition. Today we take the square roots of both sides and thus use the change of dimension meaning of powering as well. The acting across interpretation is not as well known today as it was then, and the geometric interpretation of square roots as change of dimension is seldom taught, so for many people the links between this formula and use meanings is not at all obvious.

2. The German astronomer Johannes Kepler [1571-1630] discovered that R, the average distance from any planet to the sun, and T, the length of time it takes that planet to revolve around the sun, are so related that $\frac{R^3}{T^2}$ = constant. Because T was known—from observing the heavens—for all planets, this enabled astronomers to calculate R in terms of the distance from the Earth to the sun. Mars was known to orbit the sun in about 687 days, the Earth in about 365. How many times further from the Sun is Mars than the Earth?



Answer: Let R = 1 for the Earth, Then for the Earth

$$\text{constant} = \frac{1^3}{365^2}$$

for Mars, $$\text{constant} = \frac{\text{Radius of Mars' orbit}^3}{687^2}$$

So $$\frac{1}{365^2} = \frac{\text{Radius}^3}{687^2}$$

$$\frac{687^2}{365^2} = \text{Radius}^3$$

$$\left(\frac{687^2}{365^2}\right)^{\frac{1}{3}} \approx \text{Radius}$$

$$1.52 \approx \text{Radius}$$

Mars is about half again as far from the Sun as the Earth.

Comment: Kepler discovered this relationship through painstaking observation and brilliant intuition, not using any mathematical derivations. Isaac Newton [1642-1727] derived the formula using the calculus he (Newton) had invented. The T^2 in the formula comes from acceleration, the R^3 from dividing R, related to the acceleration of the planet in circular motion, by $\frac{1}{R^2}$, related to the force of gravity. The use meanings involved are acting across multiplication and rate division (extended to acceleration), but in a calculus setting. The formula is typically derived in first-year college Physics courses. It is a classic example of a formula that people employed without anyone knowing why it works, but a formula later derived from basic principles using the most sophisticated mathematics of its time.

3. Over the past 75 years, the time t for the world's record in the mile run in the year Y has been closely approximated by the equation

$$t = -.4Y + 1020.$$

Explain why this formula cannot be true for all time.

Answer: If one substitutes the year 1980 for Y,

$$t = -.4(1980) + 1020$$

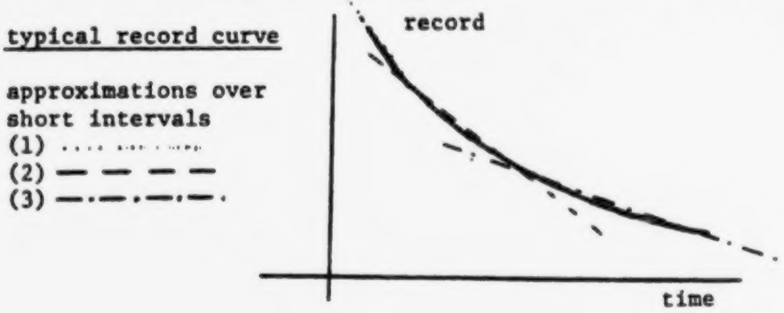
giving a time of 228 seconds, only 0.6 seconds off the record in that year (3:48.6 or 228.6 seconds). But try a year very much in the future, let us say the year 2600. The time will be

$$t = -.4(2600) + 1020 = -20 \text{ seconds}$$

which is impossible (unless by that time someone has figured out how to finish a race before one starts).

Comment: The calculation shows that the formula must be approximate. The $-.4$ is a rate of change factor, meaning that the record has over the past 75 years tended to change at a rate of .4 seconds downward per year. An equivalent formula is $t - 1950 = -.4(Y - 240)$, with the subtractions being shifts to fix an approximation of 240 seconds for the record in 1950. Thus, in this formula, the constituent operations can be explained. However the formula cannot hold forever because it represents no fundamental relationship between running and mathematics.

Comment: Over short periods of time, the change in a record can usually be approximated closely by a formula of the type in the example because a line is a good approximation to a small part of a continuous monotone curve.



Summary

Many applications of arithmetic involve more than one operation or involve the same operation used in more than one way. The purpose of this chapter has been to give a representative selection of such applications and to relate them to the use classes of the individual operations.

The examples include consumer applications involving costs, proportions, and annuities; applications of lines and slopes appropriate to topics in algebra; uses of the Pythagorean Theorem and area formulas studeid in geometry; and instances of physics not studied until calculus. They illustrate that the use classes of earlier chapters are analogous to basic properties of mathematical systems. Just as all mathematical properties of numbers can be traced back to a small number of fundamental principles, so it seems that the operations involved in most applications of arithmetic can be traced back to a small number of use classes of the operations.

Pedagogical Remarks

It is obvious that a complicated situation can often be made to feel less complicated by analyzing the constituent parts of the situation. It is less obvious that, for some children, this separation increases the perceived difficulty. Consider the following example:

> A child weighs 85 pounds on September 1st and 93 pounds on the next February 1st, 5 months later. How fast has the child gained weight?

Solving this problem requires two operations, subtraction and division. Specifically, subtract 85 pounds from 93 pounds to determine the weight gain and divide the difference, 8 pounds, by 5 months, to get $1.6 \frac{\text{lb}}{\text{mo}}$, the rate of weight gain.

In the solution, the subtraction is a comparison less familiar to most students than take-away. The division is a rate, not usually taught. For those students who understand comparison and rate, the separation into two operations is satisfying and achieves the goal of simplification. But for those students who are not familiar with comparison and rate, the separation into the two constituent operations has merely increased the task from dealing with a single problem to dealing with two problems that are not understood.

A teacher will have success by breaking down a "combined operations" problem when students are already familiar with the uses in the constituent operations that arise. It is for this reason that we spend so much time in this book on the individual operations.

Getting started. The concept of breaking down a difficult situation into simpler constituent situations is not one that comes naturally to all students. Yet it can be taught quite early, as soon as there are two operations that the student understands.

Begin with a simple one-step problem.

(1) John has 12 pieces of candy. He gives 5 pieces to Mary. How many are left?

Change one of the given pieces of information so that it is a result of some problem. For example:

(2) John found 9 pieces of candy in one box and 3 pieces in another. He gave 5 pieces to Mary. How many are left?

or (3) John found 12 pieces of candy. He gave 3 pieces to Mary and 2 to Bill. How many are left?

Both questions (2) and (3) can be answered in more than one way. In (2), one can subtract 5 from 9 and then add 3, or one can add 9 and 3 and then subtract 5. In (3), one can do either $12 - 3 = 9$ and then $9 - 2 = 7$, or one can do $3 + 2 = 5$, $12 - 5 = 7$. This flexibility is a characteristic of many problems involving more than one operation.

Ask students to make up a more complicated problem. Their ingenuity and the resulting degree of complication are often considerable.

With other operations in hand, the above questions can be made a little more complicated.

(4) In a class, 100 pieces of candy were distributed evenly among 25 students. John gave 2 from his share to Mary and 3 to Bill. How many pieces did he have left?

(5) John has a box of chocolates with 8 rows of
of chocolates each containing 9 pieces of
candy. If he gives 10 pieces to Mary and 11
to Bill, how many pieces will he have left?

The important idea here is to have students build up the complexity. When they have had some experiences doing so, they are less fearful of those problems that have built-in complexity.

The quote that begins Part II is one which has become quite famous and is too often true. That is, many students decide what to do with a problem merely by looking at the numbers in the problem and not at all by examining the context or, in some cases, not even reading the problem. A problem like (5) above, with 4 numbers of approximately the same size, is viewed as addition, and the student answers 38. Number cues do work at times--if they never worked, students would not be led to them. But they are even worse than verbal cues and the teacher must watch that operations in the problems given students are not able to be guessed by number size alone.

One way to teach students to examine context is to present several problems with the same numbers. Here is such a set, taken from Word Problems - Introductory Book, by Anita Harnadek, published by Midwest Publications. In that book all of the problems use the numbers 10 and 40.

(6) Franklin borrowed $10 from Jeffers. With interest, Franklin had to pay Jeffers a total of $40. How much was the interest?

(7) Franklin made 10 payments of $40 each to Jeffers. How much did Franklin pay Jeffers?

(8) Franklin borrowed \$40 from Jeffers. He is to pay it back in 10 equal payments. How much is each payment to be?

(9) Franklin borrowed \$40 from Jeffers. He has paid back \$10 so far. How many more \$10 payments does he have to make?

and so on.

(It may be of interest that one of the ways in which we checked the comprehensiveness of the categorization in Part II was to examine practice books of word problems to see that all the problems in them fit into one of our categories.)

Situations that are not separable. Not all situations are separable into single operations with obvious meaning, because many complicated situations utilize mathematical simplifications that disguise the original inputs into the situation. For example, if at the end of a child's 144th month (12th birthday), \$75 is put into a college account and an additional \$75 is put in every month thereafter, then the total T put in when the child is M months old is given by the formula

$$T = 75(M-144) + 75 \text{ dollars}$$

and you may be able to explain each number in the formula. But it is natural to want to "simplify" the right side to

$$T = 75M - 9525$$

and the origin of the 9525 and the meaning of the subtraction are both hidden.

Thus one should not expect to separate all situations that combine operations. But it is the case that many more of these situations are separable than most people realize.

Questions

1. A person buys 5 cans of pop reduced from 39¢ each to 29¢ each. (a) How much has the person saved? (b) The answer can be computed in two different ways. Give the two ways and name possible use classes.
2. (a) By how much more will an amount of money be multiplied if it is invested at a 10% annual rate for 5 years than if it is invested at an annual rate of 8% for 5 years? (b) Name the two use classes involved in this situation.
3. A keypuncher was able to code 1200 pieces of information in 3.5 hours. (a) At this rate how many hours will it take the keypuncher to code 4500 pieces of information? (b) What use classes are involved in your solution?
4. According to the approximation to the mile record formula in Section C, what will be the world record for the mile run in the year 2000? If you are familiar with track and field, indicate whether you believe this estimate to be high or too low? (If you are not familiar with track and field, you may wish to ask someone who is.)
5. The planet Jupiter takes about 12 years to orbit the Sun. Using Example 2, Section C, about how many times farther from the Sun is Jupiter than is the Earth?
6. Which two operations are customarily used in each of the following situations?
   - (a) finding slope
   - (b) calculating final amount on an investment yielding compound interest
   - (c) balancing a checkbook
   - (d) calculating the value of an inventory

7. At the beginning of the day, Ms. Carlson prepared 60 tests for two classes. The first period 25 tests were used, the second period 27 tests were used. The number of tests left in this situation may be considered as either (x-y) - z or x - (y+z). (a) What do x, y, and z stand for? (b) How do the use classes differ in the two expressions?

8. Shoe size S and foot length F (in inches) for men in the United States are approximately related by the formula $S = 3F - 24$. (a) How large must a foot be before this formula makes sense? (b) Give some foot lengths and the corresponding shoe size. (c) Try the formula on someone to see if it works. (d) The formula involves a multiplication and a subtraction. Can you give a use class explanation for either operation?

10. Make up an application that involves: (a) powering and subtraction; (b) powering and division; (c) two different use classes of multiplication.

Notes and Commentary

1. The uses of operations considered as a postulate set
2. Work of others
3. Meanings of relationships
4. Arithmetic vs. algebra

1. *The uses of operations considered as a postulate set*. In many probability texts (e.g., Mosteller, Rourke, & Thomas, (1972)), there are fundamental principles of addition and multiplication equivalent to the following:

> (a) Addition principle: Let $n(S)$ be the number of elements in a set S. Then if sets A and B are disjoint, $n(A \cup B) = n(A) + n(B)$.
>
> (b) Multiplication principle: Let $A \times B$ be the cartesian product of sets A and B. Then $n(A \times B) = n(A) \cdot n(B)$.

These principles act as two of the postulates for problems involved in counting. From the postulates and appropriate definitions, many theorems can be proved. For example, one can deduce that if sets A and B are *not* disjoint, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. And, since probabilities in finite sample spaces are normally defined as ratios of counts, from these postulates and properties of real numbers, the basic principles of probability can be derived.

In a similar way, the use classes of the individual operations of Chapters 5–9 form what might be considered as a postulate set for the applications of arithmetic. (The postulates for counting and probability given above are covered in two of these use classes. Principle (a) is a specialized form of putting together addition and Principle (b) is a form of the acting across use class of multiplication.) This chapter exhibits the power of this postulate set both in deducing and in explaining applications that seem more complicated.

The distinction between use classes and use meanings is that the use meanings form what might be termed a "more minimal" postulate set than the set of all use classes. From this more minimal set, the other use classes can be derived. But just as one does not normally study geometry or other mathematics from a minimal set of postulates, in this book we have made no attempt to operate from such a minimal set.

Considering the use classes as postulates in a mathematical system, applications that involve exactly two use classes (Section A) are roughly equivalent to theorems whose proofs require applying only two postulates. The applications that involve more than two operations (Section B) are akin to theorems whose proofs require more than two steps. Applications that are more

BEST COPY AVAILABLE

complicated and whose operations are not so easily analyzed by use classes (Section C) are analogous to theorems whose proofs are difficult.

2. Work of others. The process we described in Note 1 above is by no means new nor has its use been restricted to counting problems and probability. Physics courses customarily begin with certain fundamental principles, such as Newton's laws of motion. Rate, acceleration, area, and volume are explicitly related to mathematics in calculus courses. Flows of liquids are explicitly related to differential equations.

What we have tried to do is to apply this process at a more elementary level and with more breadth than is customarily the case. We have found no corresponding attempts.

The closest work to ours that we have found is that of Worth Osburn (1929). He lists "the more important types of problems and exercises of more than one step as taught in certain schools in grades 7 and 8". His list contains 57 types, of which the following are typical.

Type 15. Finding the amount of the dividends when the total value and the rate are given.
24. Changing Fahrenheit temperatures to centigrade.
55. Finding the volumes of cylinders.

Several of his types involve algorithms, not applications.

Type 20. Adding fractions with the same denominators.
47. Find the hypotenuse.

A similar list for grades 3-6 contains 112 types. In each list about 2/3 of the entries can be identified with viable applications.

Osburn invented a symbolism to describe the operations utilized in problems and exercises requiring more than one step for their solution. For instance, SM means a problem that requires first subtraction, then multiplication. One of his examples of SM is:

> "My newspaper costs ____ cents a day. How much does it cost for a week if I do not buy a paper on Sunday?" (p. 241)

He does not include powering, so his symbolism for finding the volume of a sphere is M5D, meaning that five multiplications followed by a division are needed. (Recall that the formula is $V = (4/3)(\pi r^3.)$ This symbolism provides a means by which his many types can be classified.

3. Meanings of relationships. Note 5 of the Notes and Commentary for Chapter 5 discussed five kinds of meanings of operations. Relationships have various kinds of meanings as well.

We illustrate with five corresponding kinds of meaning for the Fahrenheit-Celsius relationship (Section B, Example 3).

A. By numerical results: Give a table of corresponding F and C values. For some people, this provides the most definitive meaning.

B. By formula: The formula C = (5/9)(F - 32) provides a computation routine for getting C values from F values. This is often given as a definition of the relationship. Like algorithms, it provides a secure but sometimes unthinking means to get answers.

C. By deduction and properties: Make appropriate assumptions about the relationship, e.g., that 0°C = 32°F and 100°C = 212°F and the relationship is linear. From this deduce the formula. Some people feel that one best understands the meaning of a relationship if it is deduced from agreed-upon statements.

D. By concrete experience: Exhibit two Fahrenheit and Celsius thermometers. Look at them on various days to determine equivalent Fahrenheit and Celsius values. In this case, meaning is extracted from experience. For some people, this provides the true meaning.

E. By use meanings: Derive the relationship from more basic uses of operations. That is the method shown in this chapter. We do not claim that this is the way to understand the Fahrenheit-Celsius relationship, but feel that it is a valuable and almost always overlooked way of connecting mathematics and reality.

4. Arithmetic vs. algebra. Many schoolbooks have students practice solving problems by setting up equations to be solved. This is fundamentally different from using arithmetic without equations. Consider the following problem, much like those in standard textbooks.

> A person has $100 in a savings account. If $25 is saved each month, how long will it take the person to have $325 in the account. (Here interest is ignored.)

First we solve using arithmetic. The problem requires two operations, subtraction comparison and then rate divisor division. That is, subtraction yields $225 as the total the person needs to save. At $25 per month, it will take $\frac{\$225}{\$25/\text{month}}$ or

9 months. Virtually every teacher would have students analyze this prolem in this two-step fashion, with or without the identification of use classes.

Now we solve the same problem using algebra. An equation which could be used is:

$$100 + 25M = 325$$

Notice that the equation displays only the operations of addition and multiplication, so its *setup* uses none of the operations done by those who answer the problem without algebra. To solve, of course, one does use the same operations.

Subtract 100 from each side.

$$25M = 225$$

Divide both sides by 25.

$$M = 9$$

Perhaps it is because the setups for equations can involve different operations than the solutions that students have so much trouble setting up problems like these. A good arithmetic problem solver has to *reverse* his or her thinking to set up the same problems in algebra.

BEST COPY AVAILABLE

Summary of Part II

	Addition	Subtraction	Multiplication	Division	Powering
	BASIC USE MEANINGS				
	shift putting together	comparison take-away	size change acting across	ratio rate	change of dimension growth
	DERIVED USE CLASSES				
from related facts	from $a+b=c$ iff $a=c-b$		from $a\times b=c$ iff $a=c\div b$		
	addition from subtraction	recovering addend	rate factor	recovering factor	[recovering base]*
from definition of "inverse"	from $a-b = a + -b$		from $a \div b = a \times \frac{1}{b}$		
		subtraction shift		division size change	
from repetition	from $\underbrace{1+1+\ldots+1}_{n\ terms} = n$		from $\underbrace{m+m+\ldots+m}_{n\ terms} = n\times m$		from $\underbrace{m\times m\times\ldots\times m}_{n\ factors} = m^n$
	[shortcut counting] (forward)	[shortcut counting] (backward)	[shortcut addition]	[shortcut subtract.]	notation
from "double reverse"	from $a-b=c$ iff $a-c=b$		from $a\div b=c$ iff $a\div c=b$		
		[recovering subtrahend]		rate divisor	

*Use classes subsumed under other use classes in our scheme are given here in brackets.

BEST COPY AVAILABLE